Unbounded frames

Jean-Pierre Antoine

Institut de Physique Théorique, Université catholique de Louvain
Louvain-la-Neuve, Belgium

(Joint work with Peter Balazs and Diana Stoeva)

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Esneux

Three successive stages (ST.Ali - JPA - J-P.Gazeau, 1991-1993)

(1) Coherent states on locally compact group

$$\int_G |\eta_g\rangle\langle\eta_g| \,\mathrm{d}\nu(g) = I \quad \Leftrightarrow \quad \int_G |\langle\eta_g|\phi\rangle|^2 \,\mathrm{d}\nu(g) = \|\phi\|^2, \quad \forall\phi \in \mathcal{H}$$

- G = locally compact group, with (left) Haar measure $\mathrm{d}\nu(g)$
- $\eta \in \mathcal{H}$, a fixed vector in the Hilbert space $\mathcal{H}$
- $\eta_g = U(g)\eta$, U = strongly continuous, square integrable unitary representation of G on $\mathcal{H}$

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(2) Coherent states on homogeneous space

- H= closed subgroup of G, $X = G/H$, ν= invariant measure on X, Borel section $\sigma : G/H \to G$,
- U = unitary representation of G, square integrable modulo the subgroup H and the Borel section σ, i.e.

$$\int_X |\eta_{\sigma(x)}\rangle\langle\eta_{\sigma(x)}| \, \mathrm{d}\nu(x) = S_\sigma, \quad \eta_{\sigma(x)} = U(\sigma(x))\eta$$

converges weakly to a bounded, positive, invertible operator S_σ

$$\Longleftrightarrow \quad \int_X |\langle\eta_{\sigma(x)}|\phi\rangle|^2 \, \mathrm{d}\nu(x) = \langle\phi, S_\sigma\phi\rangle, \quad \forall\phi \in \mathcal{H}$$

(3) Reproducing triples : group structure is not needed !

Reproducing triple $\{\mathfrak{H}, \Lambda, S\}$:

- measure space (X, ν)
- bounded, positive, invertible operator S, acting on a Hilbert space $\mathcal{H}$
- ν-measurable function Λ from X into the bounded positive operators on $\mathcal{H}$, s.t. (weakly)

$$\int_X \Lambda(x)\, \mathrm{d}\nu(x) = S$$

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- $\{\mathfrak{H}, \Lambda, S\}$ = frame if
  - rank $\Lambda(x)$ is constant and finite
  - S^{-1} is a bounded operator

- Coherent states
  - X = locally compact space with measure ν
  - $\Psi := \{\psi_x,\, x \in X\} \subset \mathcal{H}$ a family of vectors indexed by points of X
  - Ψ is a set of coherent states (CS) if

$$\int_X \langle f, \psi_x \rangle \langle \psi_x, f' \rangle \, \mathrm{d}\nu(x) = \langle f, Sf' \rangle, \ \forall f, f' \in \mathcal{H}$$

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- If S^{-1} is bounded, usual frame condition : $\{\psi_x\} \subset \mathcal{H}$ is a frame if there exist constants $\mathrm{m} > 0$ and $\mathrm{M} < \infty$ such that

$$\mathrm{m}\, \|f\|^2 \leqslant \int_X |\langle\psi_x, f\rangle|^2 \,\mathrm{d}\nu(x) \leqslant \mathrm{M}\, \|f\|^2, \forall\, f \in \mathcal{H}$$

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- If S^{-1} is unbounded, with dense domain $\mathrm{Dom}(S^{-1})$, so that we can write

$$0 < \int_X |\langle\psi_x, f\rangle|^2 \, \mathrm{d}\nu(x) \leqslant \mathrm{M}\, \|f\|^2, \forall\, f \in \mathcal{H},$$

  then Ψ is called an unbounded frame

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- May be formulated in a Gel'fand triplet $\Phi \subset \mathfrak{H} \subset \Phi^\times$, in which Φ is essentially the domain of S^{-1} with graph norm

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- Since $S > 0$, W_Ψ is injective and $W_\Psi^{-1} : \mathrm{Ran}(W_\Psi) :\to \mathcal{H}$ is well-defined
- $\mathrm{Ran}(W_\Psi)$ is a closed subspace $\mathfrak{H}_\Psi$ of $L^2(X, \mathrm{d}\nu)$, which is complete for the new scalar product

$$\langle \phi, \phi' \rangle_\Psi := \langle \phi, W_\Psi \, S^{-1} \, W_\Psi^{-1} \phi' \rangle_{L^2(X)}, \;\; \phi, \phi' \in \mathrm{Ran}(W_\Psi)$$

  and $W_\Psi : \mathcal{H} \to \mathfrak{H}_\Psi$ is unitary:

$$\begin{aligned}
\langle \phi, \phi' \rangle_\Psi &= \langle W_\Psi f, W_\Psi f' \rangle_\Psi = \langle W_\Psi f, W_\Psi \, S^{-1} \, W_\Psi^{-1} W_\Psi f' \rangle_{L^2(X)} \\
&= \langle W_\Psi f, W_\Psi \, S^{-1} \, f' \rangle_{L^2(X)} \\
&= \langle f, W_\Psi^* W_\Psi \, S^{-1} \, f' \rangle_{\mathcal{H}} \\
&= \langle f, f' \rangle_{\mathcal{H}}
\end{aligned}$$

- The projection from $L^2(X, \mathrm{d}\nu)$ onto $\mathfrak{H}_\Psi$ is $\mathbb{P}_\Psi = W_\Psi {W_\Psi}^*$ and it is an integral operator with kernel $K(x, y) = \langle \psi_x, S^{-1}\psi_y \rangle$
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- Inverting W_Ψ on its range by the adjoint operator, one gets a reconstruction formula

$$f = W_\Psi^{-1}\phi = W_\Psi^*\phi = \int_X \phi(x)\, S^{-1}\, \psi_x \; \mathrm{d}\nu(x), \;\; \phi \in \mathfrak{H}_\Psi$$

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  Then one has :

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{W_\Psi} & R_W \subset & \overline{R_W} \subset L^2(X, \mathrm{d}\nu) \\ \cup & & \cup & \\ \mathrm{Dom}(S^{-1}) = R_S & \xrightarrow{W_\Psi} & W_\Psi(R_S) \subset L^2(X, \mathrm{d}\nu) & \end{array}$$

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- Define the Hilbert space $\mathfrak{H}_\Psi := \overline{W_\Psi(R_S)}^{\Psi}$
  (completion in norm $\|\cdot\|_\Psi$)

  Then the map W_Ψ, restricted to the dense domain $\mathrm{Dom}(S^{-1}) = R_S$, is an isometry into $\mathfrak{H}_\Psi$:

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- Thus W_Ψ extends by continuity to a unitary map from $\mathcal{H}$ onto $\mathfrak{H}_\Psi := \overline{W_\Psi(R_S)}^{\Psi}$

- Thus we get $\mathfrak{H}_\Psi = R_W$, which therefore is a subspace (though not necessarily closed) of $L^2(X, \mathrm{d}\nu)$:

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$\Rightarrow$ S_Ψ and S_Ψ^{-1} are unitary images of S and S^{-1}, thus

$$\|S_\Psi\|_\Psi = \|S\|_{\mathcal{H}}$$

- Definition : the unbounded frame $\Psi = \{\psi_x,\, x \in X\}$ is regular if $\psi_x \in \mathrm{Dom}(S^{-1}),\ \forall\, x \in X$

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- If Ψ is regular, we get the same reconstruction formula

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- If Ψ is not regular, use language of distributions :

  $K(x,y)$ defines a bounded sesquilinear form over $\mathfrak{H}_\Psi$

- Best formulation : in terms of a Gel'fand triplet

$$\mathfrak{H}_\Psi \;\subset\; \mathfrak{H}_0 \;\subset\; \mathfrak{H}_\Psi^\times$$

  where

  - $\mathfrak{H}_0 := \overline{\mathfrak{H}_\Psi} = \overline{R_W} =$ closure of $\mathfrak{H}_\Psi$ in $L^2(X, \mathrm{d}\nu)$
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$\Rightarrow$ $\mathfrak{H}_\Psi^\times$ carries the unbounded version of the dual frame

- Even if Ψ is not regular, one has

$$\iint_{X \times X} \overline{\phi(x)} K(x,y) \chi(y) \; \mathrm{d}\nu(x) \; \mathrm{d}\nu(y) = \langle W_\Psi^{-1} \phi, S W_\Psi^{-1} \chi \rangle_{\mathcal{H}}$$

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- Since W_Ψ is an isometry and S is bounded, this relation defines a bounded sesquilinear form over $\mathfrak{H}_\Psi$:

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- Let $\mathfrak{H}_\Psi^\times$ = completion of $\mathfrak{H}_\Psi$ in the norm given by K^Ψ
- Reproducing property of $K(x,y)$ implies

$$\int_X \overline{\phi(x)}\,\chi(x)\, \mathrm{d}\nu(x) = \langle \phi,\chi\rangle_{L^2(X,\,\mathrm{d}\nu)} = K^\Psi(\phi,\chi)$$

Thus, with continuous and dense range embeddings,

$$\mathfrak{H}_\Psi \;\subset\; \mathfrak{H}_0 \;\subset\; \mathfrak{H}_\Psi^\times$$

where

- $\mathfrak{H}_\Psi = R_W$ = Hilbert space for the norm $\|\cdot\|_\Psi = \langle \cdot, W_\Psi\, S^{-1}\, W_\Psi^{-1} \cdot\rangle^{1/2}$
- $\mathfrak{H}_0 = \overline{\mathfrak{H}_\Psi}$ is the closure of $\mathfrak{H}_\Psi$ in $L^2(X,\,\mathrm{d}\nu)$
- $\mathfrak{H}_\Psi^\times$ = completion of $\mathfrak{H}_0$ in the norm $\|\cdot\|_\Psi^\times := \langle \cdot, W_\Psi\, S\, W_\Psi^{-1} \cdot\rangle^{1/2}$
  = conjugate dual of $\mathfrak{H}_\Psi$

- $\mathfrak{H}_\Psi^\times$ = conjugate dual of $\mathfrak{H}_\Psi$:
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  - Inner product $\langle X_\phi, X_\chi \rangle_{\Psi^\times} = \langle W_\Psi^{-1}\phi, SW_\Psi^{-1}\chi \rangle_{\mathcal{H}}$ + completion
    $\Rightarrow$ Hilbert space $\mathfrak{H}_\Psi^\times$

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- One has also, for each $X \in \mathfrak{H}_\Psi^\times$,

$$X(\phi) = \langle X, X_\phi \rangle = \langle X, K^\Psi(\phi, \cdot) \rangle_{\Psi^\times}$$

which expresses the reproducing property of the kernel K^Ψ as a function over $\overline{\mathfrak{H}_\Psi} \times \mathfrak{H}_\Psi$

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- Note : same definitions hold if Ψ is only a Bessel sequence

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Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}$, with analysis operator $C : \mathcal{H} \to \ell^2$, synthesis operator $D : \ell^2 \to \mathcal{H}$ and frame operator $S : \mathcal{H} \to \mathcal{H}$. Then:

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(5) *C is unitary as operator on $\mathfrak{H}_\Psi$, and so can be inverted on its range by its adjoint, to get the reconstruction formula*

$$f = S^{-1}DCf = \sum_k \langle \psi_k, f\rangle S^{-1}\psi_k, \quad \text{for every } f \in \mathcal{H}$$

Let Ψ be an unbounded frame :

$$0 < \sum_{n\in\Gamma} |\langle \psi_n, f\rangle|^2 \leqslant \mathsf{M}\, \|f\|^2, \forall\, f \in \mathcal{H},\ f \neq 0$$

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- G^{-1} is positive, since its inverse G is bounded and thus the spectrum of G^{-1} is bounded away from 0.

- Putting everything together, we have the following diagram :

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(1) *$\mathfrak{H}_\Psi$ coincides with R_C and $C : \mathcal{H} \to \mathfrak{H}_\Psi$ is an isomorphism (unitary map).*

(2) *The norm $\|.\|_\Psi$ is equivalent to the graph norm of $G^{-1/2}$ and, therefore,* $\mathrm{Dom}(G^{-1/2}) = \mathfrak{H}_\Psi$.

(3) *$C : \mathcal{H} \to \mathfrak{H}_\Psi$ can be inverted on $\mathfrak{H}_\Psi$ by its adjoint $C^{*(\Psi)} = S^{-1}D\restriction\mathfrak{H}_\Psi$, which yields the following reconstruction formula, for every $f \in R_S$,*

$$f = C^{*(\Psi)}Cf = \left(S^{-1}D\right)Cf$$

(4) *For all $f \in R_S$, we also have*

$$f = \sum_k \langle \psi_k, C^{*^{(\Psi)}} G^{-1} Cf \rangle \, \psi_k$$

(5) *For all $f \in R_D$, we have the alternative reconstruction formula*

$$f = \sum_k \left[G^{-1} \left(\langle \psi_k, f \rangle_{\mathcal{H}} \right) \right] \psi_k$$

- Exactly as in the continuous case, we have the following diagram:

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\begin{array}{ccccc}
\mathcal{H} & \xrightarrow{C} & \mathfrak{H}_\Psi = R_C \subset \overline{R_C} \subset \ell^2 \\
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Corollary

$G^{1/2} : \overline{R_C} \to \mathfrak{H}_\Psi$ is an isomorphism and so is its inverse $G^{-1/2} : \mathfrak{H}_\Psi \to \overline{R_C}$.

- In order to get a nice reproducing kernel, we have to assume Ψ to be regular. Indeed:

Theorem

Let (ψ_k) be a regular unbounded frame. Then $\mathfrak{H}_\Psi$ is a reproducing kernel Hilbert space, with kernel given by the operator $S^{-1}D$, which is a matrix operator, given by the matrix $\mathcal{G}$, where

$$\mathcal{G}_{k,l} = \langle \psi_k, S^{-1}\psi_l \rangle = \langle \psi_k, C^{-1}G^{-1}C\psi_l \rangle.$$

Proof : Let $\phi = Cf \in \mathfrak{H}_\Psi$. Then,

$$\sum_l G_{k,l}\phi_l = \sum_l \langle \psi_k, S^{-1}\psi_l\rangle \phi_l = \langle \psi_k, S^{-1}\sum_l \psi_l \phi_l\rangle = \langle \psi_k, S^{-1}D\phi\rangle$$
$$= (CS^{-1}D\phi)_k = (CS^{-1}DCf)_k = (Cf)_k = \phi_k.$$

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- Conclusion : everything works as usual, including reconstruction, provided Ψ is regular, i.e. $\psi_n \in \mathrm{Dom}(S^{-1}),\ \forall\, n$

- Let (e_k) be an ONB in $\mathcal{H}$ with index set $\mathbb{N}$. Let $\psi_k = \frac{1}{k} e_k$. Then (ψ_k) is an unbounded frame :

$$0 < \sum_k |\langle \psi_k, f \rangle|^2 \leqslant \sum_k |\langle e_k, f \rangle|^2 = \|f\|^2$$

The lower bound is 0, since for $f = e_p$, one has $\sum_k |\langle \psi_k, f \rangle|^2 = \frac{1}{p^2}$

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- Let $\phi_k = k\, e_k$: the sequence (ϕ_k) is dual to (ψ_k), since one has

$$\sum_k \langle \psi_k, f \rangle \phi_k = f$$

  This is the unbounded dual frame, living in $\mathfrak{H}_\Psi^\times$

- In this case, the frame operator is $S = \mathrm{diag}(\frac{1}{k})$ and $S^{-1} = \mathrm{diag}(k)$, so that the inner products are, respectively :
  - For $\mathfrak{H}_\Psi$: $\langle c, d\rangle_\Psi = \sum_k k\, \overline{c_k}\, d_k$
  - For $\mathfrak{H}_0$: $\langle c, d\rangle_0 = \sum_k \overline{c_k}\, d_k$
  - For $\mathfrak{H}_\Psi^\times$: $\langle c, d\rangle_\Psi^\times = \displaystyle\sum_k \frac{1}{k}\, \overline{c_k}\, d_k$

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- Generalization :
  - $(m_n e_n)$, where $m \in \ell^\infty$ has a subsequence converging to zero and $m_n \neq 0, \forall n$: an unbounded frame, not a frame
  - $(\frac{1}{m_n} e_n)$: satisfies the lower frame condition, but is not Bessel.

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Duality - 1

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- In the unbounded case :

Lemma

Let Φ be a Bessel sequence in $\mathcal{H}$. If there exists Ψ such that (at least) one of the following three conditions hold:

(a$_1$) $\sum_n \langle \psi_n, f \rangle \, \phi_n = f, \forall \, f \in \mathcal{H}$

(a$_2$) $\sum_n \langle \phi_n, f \rangle \, \psi_n = f$ *with unconditional convergence of the series for every* $f \in \mathcal{H}$

(a$_3$) $\sum_n \langle \phi_n, f \rangle \, \psi_n = f, \forall \, f \in \mathcal{H}$, *and* $\sum_n \langle \psi_n, f \rangle \, \phi_n$ *converges for all* $f \in \mathcal{H}$

then Φ is an unbounded frame for $\mathcal{H}$ and Ψ satisfies the lower frame condition.

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Duality - 2

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  - 'Dual' $\widetilde{\Psi}$ = sequence satisfying the lower frame condition

    $\Rightarrow S$ unbounded, S^{-1} bounded
- From exact results, there is duality between
  - Unbounded frames = complete Bessel sequences
  - Complete sequences satisfying the lower frame condition

$\Rightarrow$ various series expansions, with appropriate convergence

- Generalization : Rank n frames
  - A set of vectors $\eta_x^i \in \mathfrak{H}, \quad i = 1, 2, \ldots, n < \infty, \quad x \in X$, is a rank n frame $\mathcal{F} = \mathcal{F}\{\eta_x^i, S, n\}$ if
    (i) for all $x \in X, \quad \{\eta_x^i, i = 1, 2, \ldots, n\}$ is a linearly independent set
    (ii) there exists a positive operator $S \in GL(\mathfrak{H})$ such that, with weak convergence,

$$\sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| \; \mathrm{d}\nu(x) := \int_X \Lambda(x) \; \mathrm{d}\nu(x) = S$$

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- Connection with frame multipliers ?

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