Baire generic histograms of wavelet coefficients and large deviation formalism in Besov and Sobolev spaces

MOURAD BEN SLIMANE

College of Sciences. King Saud University

Plan presentation

1. Introduction of the problem.
2. Wavelet characterizations of Besov and Sobolev spaces.
3. Exact definitions of histograms of wavelet coefficients.
4. Heuristic derivation of the L. D. Formalism.
5. First Results.
6. Proofs.
7. Second Results.
8. Proofs.

Let $f : \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{C}$ be periodic over $\mathbb{Z}^d$. Let $\alpha > 0$, $x_0 \in \mathbb{T}$ the torus $\mathbb{R}^d/\mathbb{Z}^d$.

$$f \in C^\alpha(x_0) \Leftrightarrow \exists P \;\; d^\circ P \leq [\alpha] \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \, .$$

$$\alpha_f(x_0) = \sup\{\alpha \;:\; f \in C^\alpha(x_0)\} \in \bar{\mathbb{R}}_+ \quad \text{Hölder exponent at } x_0 \, .$$

$$d_f \colon \alpha \mapsto dim(\{x \;:\; \alpha_f(x) = \alpha\}) \quad \text{Hölder spectrum} \quad .$$

Conjecture : Large Deviation Formalism

If $f \in C^\gamma(\mathbb{T})$ for $\gamma > 0$ then $d_f = \rho_f$.

The wavelet density $\rho_f(\alpha)$: (when $j \to \infty$) there are $\sim 2^{j\rho_f(\alpha)}$ wavelet coefficients $|C_{j,k}|$ of f of size $\sim 2^{-\alpha j}$.

B. S : J. Math. Anal. Appl. **349** (2009) 403-412, we proved the L. D. Formalism for quasi-all functions, in the sense of Baire, i.e., on a countable intersection of open dense sets, (G_δ set) of the Banach spaces $L^{p,s}(\mathbb{T})$ and $B_p^{s,q}(\mathbb{T})$ for $s > d/p$.

We also study it in the Baire's vector space $V = \bigcap_{\varepsilon>0,p>0} B_p^{s(\frac{1}{p})-\frac{\varepsilon}{p},p}$ where $s : q \mapsto s(q)$ is a C^1 and concave function on $\mathbb{R}^+$ satisfying $0 \leq s' \leq d$ and $s(0) > 0$.

$$p > 1 \ \forall\, s > 0 \qquad L^{p,s} = L^{p,s}(\mathbb{T}) := \{f \in L^p \; ; \; (-\Delta)^{s/2} f \in L^p\}$$

$$B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty} \tag{1}$$

$$B_p^{s-\varepsilon,q} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s+\varepsilon,q} \; . \tag{2}$$

Without any loss of generality, take $d = 1$ and the Meyer's $\psi \in S(\mathbb{R})$.
The functions 1 and $2^{j/2}\psi_{j,k}(x) := 2^{j/2} \sum_{l \in \mathbb{Z}} \psi(2^j(x-l)-k)$,

$j \geq 0, \ \ k \in \{0, \cdots, 2^j - 1\}$, form an orthonormal basis of $L^2(\mathbb{T})$.

$$f \in L^2(\mathbb{T}) \Rightarrow f = \int_{[0,1]} f(t)\, dt + \sum_{j,k} C_{j,k}\psi_{j,k} \ .$$

$$C_{j,k} = C_{j,k}(f) = 2^j \int_{[0,1]} f(t)\overline{\psi_{j,k}}(t)\, dt \ .$$

$$f \in B_p^{s,q} \Longleftrightarrow \left(\sum_j \left(\sum_k |C_{j,k} 2^{(s-\frac{1}{p})j}|^p \right)^{q/p} \right)^{1/q} < \infty \qquad (3)$$

$$f \in L^{p,s}(\mathbb{T}) \Longleftrightarrow \left(\sum_j \sum_k |C_{j,k}|^2 2^{2sj} 1_{[k2^{-j},(k+1)2^{-j}[}(x) \right)^{1/2} \in L^p \qquad (4)$$

Let $\alpha \in \mathbb{R}$, for $j \geq 0$ let
$N_j(\alpha) = Card\{k \in \{0, \cdots, 2^j - 1\} \; ; \; |C_{j,k}| \geq 2^{-\alpha j}\}$.
The wavelet profile ν_f : $\alpha \mapsto \lim_{\varepsilon \searrow 0^+} \left(\limsup_{j \to \infty} \left(\frac{\log N_j(\alpha + \varepsilon)}{\log(2^j)} \right) \right)$.

The wavelet density ρ_f :
$\alpha \mapsto \inf_{\varepsilon > 0} \left(\limsup_{j \to \infty} \left(\frac{\log(N_j(\alpha + \varepsilon) - N_j(\alpha - \varepsilon))}{\log(2^j)} \right) \right)$.
At scale j (when $j \to \infty$) : there are about $2^{\nu_f(\alpha) j}$ (resp. $2^{\rho_f(\alpha) j}$) wavelet coefficients of size $|C_{j,k}| \geq 2^{-\alpha j}$ (resp. $\sim 2^{-\alpha j}$).

ν_f is nondecreasing, ν_f and ρ_f take their values in $\{-\infty\} \cup [0, d = 1]$ and $\rho_f \leq \nu_f$.

Unlike ρ_f the function ν_f does not depend on the chosen wavelet basis.

Jaffard : If $f \in C^\gamma$ for $\gamma > 0$ then

$$\alpha_f(x) = \liminf_{j\to\infty} \inf_k \left(\frac{\log(|C_{j,k}|)}{\log(2^{-j} + |k2^{-j} - x|)} \right) . \tag{5}$$

Heuristic arguments :
If $[k2^{-j}, (k+1)2^{-j}[$ contains x with $\alpha_f(x) = \alpha$ then $|C_{j,k}| \simeq 2^{-\alpha j}$.
So we expect to find $2^{jd_f(\alpha)}$ such coefficients, hence $d_f(\alpha) = \rho_f(\alpha)$.

Theorem

$\forall f \in B_p^{s,q}(\mathbb{T})$ (resp. $L^{p,s}(\mathbb{T})$)

$$\forall\, \alpha \qquad \rho_f(\alpha) \le \nu_f(\alpha) \le \alpha p - sp + 1\ . \tag{6}$$

$$(Jaffard): If \quad s > \frac{1}{p}\ then\ \forall\, \alpha \ge s - \frac{1}{p} \qquad d_f(\alpha) \le \alpha p - sp + 1\ . \tag{7}$$

Generically (in the sense of Baire), if $f \in B_p^{s,q}(\mathbb{T})$ (resp. $L^{p,s}(\mathbb{T})$) then ρ_f and ν_f are finite in $[s - \frac{1}{p}, s]$ on which

$$\nu_f(\alpha) = \rho_f(\alpha) = \alpha p - sp + 1\ . \tag{8}$$

(Jaffard) : If $s > 1/p$, then generically in $B_p^{s,q}(\mathbb{T})$ (resp. $L^{p,s}(\mathbb{T})$)

$$\forall\, \alpha \in [s - \frac{1}{p}, s] \qquad d_f(\alpha) = \alpha p - sp + 1\ . \tag{9}$$

Case $p < \infty, q < \infty$

We should first find a specific "saturating wavelet series" F.

Then using the separability of $B_p^{s,q}$ we will generate a dense G_δ-set (i.e., a countable intersection of dense open sets).

Saturating wavelet series

Let $j \geq 1$ and $0 \leq k \leq 2^j - 1$. Consider the irreductible representation

$$\frac{k}{2^j} = \frac{K}{2^J} \qquad \text{where} \quad K \in \mathbb{Z} - (2\mathbb{Z})\ . \tag{10}$$

Let

$$F = \sum_{j \geq 1, k} C_{j,k} \psi_{j,k} \ \text{ where } C_{j,k} = \frac{1}{j^a} 2^{-(s-\frac{1}{p})j} 2^{-\frac{1}{p}J} \ \text{ and } a = \frac{2}{p} + \frac{2}{q} + 1\ .$$

$0 \leq J \leq j \Rightarrow \forall (j,k) \qquad \frac{1}{j^a} 2^{-sj} \leq |C_{j,k}| \leq \frac{1}{j^a} 2^{-(s-\frac{1}{p})j}$ and both the left and right terms are attained then ρ_F is defined in $[s - \frac{1}{p}, s]$.

For each $1 \leq J \leq j$ there are $\frac{2^J}{2}$ values of k satisfying (10)
$\Rightarrow F \in B_p^{s,q}(\mathbb{T})$ and on $[s - \frac{1}{p}, s] \quad \nu_F(\alpha) = \rho_F(\alpha) = \alpha p - sp + 1$.

The dense G_δ-set

$p < \infty$ and $q < \infty \Rightarrow B_p^{s,q}$ is separable $\Rightarrow \exists\ (f_n)$ dense in $B_p^{s,q}$.
Let

$$g_n = \sum_{j<n} \sum_{k=0}^{2^j-1} C_{j,k}(f_n)\ \psi_{j,k} \ + \ \sum_{j\geq n} \sum_{k=0}^{2^j-1} C_{j,k}(F)\ \psi_{j,k}\ .$$

(g_n) is dense in $B_p^{s,q}$ and are saturating functions.

We set

$$A = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B(g_n, r_n) \text{ where } r_n = \frac{1}{2n^a} 2^{-n/p}\ .$$

A is a countable intersection of dense open sets in $B_p^{s,q}$

$$f \in A \Rightarrow \forall m\ \exists n = n_m \geq m\ ;\ \forall k \quad \frac{1}{2}|C_{n_m,k}(F)| \leq |C_{n_m,k}(f)| \leq 2|C_{n_m,k}(F)|\ .$$

We deduce that $\rho_f \geq \rho_F$ and so $\rho_f = \rho_F$.

Case p and/or $q = \infty$

We only consider the case where $p = q = \infty$, i.e. $C^s(\mathbb{T})$ witch is not separable, the argument in this case is slightly different from the previous one. The proof in the case where only one among p and q is equal to ∞ is similar.

For $n \in \mathbb{N}$ set

$$E_n = \{g \in C^s \; ; \; \forall (j,k) \;\; \exists M \in \mathbb{Z}^* \;\; C_{j,k}(g) = M 2^{-n} 2^{-sj}\} \; .$$

Lemma

$\forall \; m \in \mathbb{N} \quad D_m := \bigcup_{n \geq m} E_n$ is dense in C^s.

We set

$$\mathcal{A} = \bigcap_m \bigcup_{n \geq m} (E_n + B(0, \frac{1}{2} \, 2^{-n})) \; .$$

$\mathcal{A}$ is a countable intersection of dense open sets in C^s.

$$f \in \mathcal{A} \Rightarrow \forall m \, \exists n_m \geq m \;\; \forall (j,k) \quad \frac{1}{2} \, 2^{-n} 2^{-sj} \leq |C_{j,k}(f)| \leq C 2^{-sj} \; .$$

We deduce that $\rho_f(\alpha) = 1 = d_f(\alpha)$ if $\alpha = s$, and $-\infty$ else.

The Sobolev case

Clearly the case $p = 2$ was proved since $L^{2,s} = B_2^{s,2}$. The proof for $p \neq 2$ follows immediately from the embeddings (2) and the proof in the case $B_p^{s,q}$ with $0 < q < \infty$.

Baire's vector space $V = \bigcap_{\varepsilon>0, p>0} B_p^{s(\frac{1}{p})-\frac{\varepsilon}{p}, p}$ where $s : q \mapsto s(q)$ is a C^1 and concave function on $\mathbb{R}^+$ satisfying $0 \leq s' \leq d$ and $s(0) > 0$.

For $0 < p < \infty$, let $q = 1/p$ and

$$s_f(q) = \sup\{s; \quad f \in B_p^{s,p}\} \ . \tag{11}$$

Using Besov embeddings, s_f is increasing and concave on $]0, \infty[$ and its right and left derivatives belong to L^∞ and satisfy

$$(s_f')_r(q) \leq d \ \text{ and } \ (s_f')_l(q) \leq d \ . \tag{12}$$

s_f is differentiable almost everywhere in $]0, \infty[$ since it is increasing.

Theorem

$$\forall f \in V \quad \forall\, \alpha \qquad \rho_f(\alpha) \leq \nu_f(\alpha) \leq \alpha p - \eta(p) + d \quad \text{where } \eta(p) = ps(1/p)\ .$$

Generically, if $f \in V$, then ρ_f is finite in $\left[s(0), \eta'(0^+) = \lim_{q\to\infty} (s(q) - qs'(q)) \right)$ on which

$$\nu_f(\alpha) = \rho_f(\alpha) = \inf_{p>0} (\alpha p - \eta(p) + d)\ .$$

(Jaffard) : Generically, if $f \in V$, then d_f is finite in $[s(0), dq_c]$ on which $d_f(\alpha) = \inf_{p\geq p_c} (\alpha p - \eta(p) + d)$ (where $p_c = 1/q_c$).

s is concave with $0 \leq s' \leq d$, there exists q_c, such that if $q < q_c$, $s(q) > dq$ and if $q > q_c$, $s(q) < dq$

The generic spectrum is composed of two parts :

- $\alpha < \eta'(p_c)$: infimum attained for $p > p_c$ and $d_f(\alpha) = \inf_{p>0}(\alpha p - \eta(p) + d)$,
- $\eta'(p_c) \leq \alpha \leq dq_c$: infimum attained for $p = p_c$ and $d_f(\alpha) = \alpha p_c$.

$s(0) \leq \eta'(p_c) \leq \eta'(0^+)$.

If $s'(\infty) = 0$ then $dq_c \leq \eta'(0^+)$.
At $\alpha = \eta'(p_c)$ the slope of the tangent to the concave function ρ_f is p_c.

We conclude that the L. D. formalism holds (resp. may fail) generically in V for $\alpha \in [s(0), \eta'(p_c)]$ (resp. $\alpha \in [\eta'(p_c), \eta'(0^+)\ ]$).