Some prevalent results about monoHolder functions

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Question

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The problem is open ....even in the case of the celebrated Weierstrass function W_H defined on $U = (0, 1)$ as

$$W_H(x) = \sum_{n \geq 0} 2^{-nH} \cos(2^n \pi x)$$

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If for some $\alpha \in (0,1)$, $f \in \mathcal{C}^\alpha(U,\mathbb{R})$ then

$$\dim_{\mathcal{H}}(\Gamma(f,U)) \leq d + 1 - \alpha\ .$$

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Can we define a notion of irregularity leading to a lower bound of $\Gamma(f, I)$?

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- Let $x_0 \in U$. The locally bounded function f belongs to $I^\alpha(x_0)$ if

$$\exists C_{x_0}, r_0(x_0) > 0, \forall r \leq r_0(x_0), \sup_{|x-x_0|\leq r} |f(x) - f(x_0)| \geq C_{x_0} r^\alpha .$$

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- The bounded function f belongs to $I^\alpha(U)$ if

$$\exists C, r_0 > 0,\ \forall x_O \in U,\ \forall r \leq r_0,\ \sup_{x\in U, |x-x_0|\leq r} |f(x) - f(x_0)| \geq C r^\alpha .$$

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Two classical examples of strongly monoHölder functions

- The Weierstrass function W_H is strongly monoHölder of exponent H.

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Two classical examples of strongly monoHölder functions

- The Weierstrass function W_H is strongly monoHölder of exponent H.
- Fractional Brownian Motion $\{B_H(t)\}_{t\in\mathbb{R}}$ is strongly monoHölder of exponent H.

The problem
The case of the box dimension

If $f \in I^\alpha(U)$ we can give a lower bound of the box dimension of $\Gamma(f, U)$

$$\dim_B(\Gamma(f, U)) \geq d + 1 - \alpha \, .$$

This inequality is false in general if we replace the box dimension with the Hausdorff dimension.
Nevertheless it is satisfied by most of the studied strongly monoHölder models.

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Two natural questions

- Can we give a lower bound of $\dim_{\mathcal{H}}(\Gamma(f,U))$ for "almost every function" of $\mathcal{C}^\alpha(\mathbb{R}^d)$?

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- Can we give a lower bound of $\dim_{\mathcal{H}}(\Gamma(f,U))$ for "almost every function" of $\mathcal{C}^\alpha(\mathbb{R}^d)$?
- Is "almost every function" of $\mathcal{C}^\alpha(\mathbb{R}^d)$ strongly monoHölder of exponent α ?

The concept of prevalence

A natural extension of "almost everywhere" in Banach spaces

In $\mathbb{R}^d$, a property holds almost everywhere if the set of points where it is not true is of vanishing Lebesgue measure.

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  - Invariance with respect to dilatation and translations.
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- These properties are a consequence of the σ–finiteness and translation–invariance of this measure.
- Unfortunately, there is no such measure in infinite dimensional spaces .

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A suitable concept of "almost everywhere" has to be defined using another approach.

In $\mathbb{R}^d$, a Borel set B has Lebesgues measure zero if and only if there exists a compactly supported probability measure μ such that,

$$\forall x \in \mathbb{R}^d, \quad \mu(x+B) = 0.$$

The idea of Christensen (1972)

In a Banach space E, a Borel set $B \subset E$ is Haar-null if there exists a compactly supported measure μ on E such that

$$\forall x \in E, \quad \mu(x+B) = 0.$$

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Definition

A subset S of E is Haar-null if it is included in a Haar-null Borel set.

Definition

The complement of a Haar-null set is called a prevalent set.

Main results

Using the concept of prevalence we can state our two main results

Theorem 1

For any $\alpha \in (0,1)$, the space $SM^\alpha(\mathbb{R}^d)$ is a prevalent subset of $\mathcal{C}^\alpha(\mathbb{R}^d)$.

Theorem 2

For any $\alpha \in (0,1)$ and for any f in a prevalent subset of $\mathcal{C}^\alpha(\mathbb{R}^d)$

$$\dim_{\mathcal{H}}(\Gamma(f,U)) = d + 1 - \alpha \, .$$

Proof of Theorem 1

Two intermediate results

- A sufficient condition on wavelet coefficients for a function f to be uniformly irregular.

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- A sufficient condition on wavelet coefficients for a function f to be uniformly irregular.
- A general technique for proving prevalent result : the technique of stochastic process.

Proof of Theorem 1

First ingredient of the proof : wavelets

Suppose we are dealing with compactly supported wavelets

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Notation : wavelet leaders

For any dyadic cube λ, set

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Proposition

Let $\alpha \in (0,1)$. If there exist $C_1, C_2 > 0$ such that for any λ of scale j,

$$C_1 2^{-j\alpha} \leq d_\lambda \leq C_2 2^{-j\alpha},$$

then f is strongly monoHölder of exponent α.

Proof of Theorem 1

Second ingredient of the proof : the stochastic process technique

- A random element X on a Banach space E is a measurable mapping X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in E.

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The stochastic process technique

Set $\mu = \mathbb{P}_X$ in the definition of a Haar-null set. Then to prove that a set A is Haar-null, it is sufficient to find some random element X on E such that

$$\forall f \in E, \quad \mathbb{P}_X(A + f) = 0.$$

Proof of Theorem 1

Prevalent behavior of functions of $C^\alpha(\mathbb{R}^d)$

Proposition

For f in a prevalent subset of $\mathcal{C}^\alpha(\mathbb{R}^d)$, there exists $C_0 > 0$ such that for any dyadic cube λ of scale j

$$|d_\lambda| \geq C_0 2^{-j\alpha}.$$

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Proof

Let $(n_{j,k}^{(i)})_{i,j,k}$ be i.i.d. Bernoulli random variables and

$$X(x) = \sum_{i=1}^{2^d-1} \sum_{j\geq 0} \sum_{|k|\leq 2^{jd}} (-1)^{n_{j,k}^{(i)}} 2^{-\alpha j} \psi_\lambda(x).$$

We apply the stochastic process technique with X as random element on $\mathcal{C}^\alpha(\mathbb{R}^d)$.

Proof of Theorem 2

A lower bound of the Hausdorff dimension of the graph of random wavelet series

Proposition (Roueff, 2003)

Let X be the following random wavelet series

$$X(x) = \sum_{\lambda} c_\lambda \psi_\lambda(x),$$

where c_λ are independent centered Gaussian random variables with standard deviation σ_λ. Define

$$s = \limsup_{J\to\infty} \liminf_{j\to\infty} (-j)^{-1} \log_2 \min_{j\le l\le j+J} \sum_k \min(1, \frac{2^{-l}}{\sqrt{2\pi}\sigma_\lambda})2^{-2l} .$$

Then almost surely $\dim_{\mathcal{H}} \Gamma(X+f, I) \geq s$.

Proof of Theorem 2

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Let $(\xi_{j,k}^{(i)})_{i,j,k}$ be i.i.d. standard Gaussian random variables. We consider the following Gaussian field

$$X(x) = \sum_{i=1}^{2^d-1} \sum_{j \geq 0} \sum_{|k| \leq 2^{jd}} \frac{\xi_{j,k}^{(i)}}{j^2 \sqrt{\log j}} \, 2^{-\alpha j} \psi_{j,k}^{(i)}(x).$$

We apply the stochastic process technique with X as random element on $C^\alpha(\mathbb{R}^d)$.

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