Some prevalent results about monoHolder functions

Marianne CLAUSEL–Samuel NICOLAY
Let $U$ an open subset of $\mathbb{R}^d$.

**Question**
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The problem is open ....even in the case of the celebrated Weierstrass function $W_H$ defined on $U = (0, 1)$ as

$$W_H(x) = \sum_{n \geq 0} 2^{-nH} \cos(2^n \pi x)$$
The problem

An upper bound

The Hausdorff dimension $\dim_{\mathcal{H}}(\Gamma(f, U))$ of $\Gamma(f, U)$ can be related to the smoothness of $f$. 
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Definition of $C^\alpha(\mathbb{R}^d)$

The function $f$ belongs to $C^\alpha(\mathbb{R}^d)$ if

$$\exists C > 0 \forall (x, y) \in U^2, \ |f(x) - f(y)| \leq C|x - y|^\alpha$$
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If for some $\alpha \in (0, 1)$, $f \in C^\alpha(U, \mathbb{R})$ then

$$\dim_{\mathcal{H}}(\Gamma(f, U)) \leq d + 1 - \alpha.$$
The problem
Onward to a lower bound

Can we define a notion of irregularity leading to a lower bound of $\Gamma(f, I)$?

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Definition of $I^\alpha(\mathbb{R}^d)$

Let $\alpha \in (0, 1)$. 

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Onward to a lower bound

Can we define a notion of irregularity leading to a lower bound of \( \Gamma(f, I) \)?

**Definition of \( I^\alpha(\mathbb{R}^d) \)**

Let \( \alpha \in (0, 1) \).

- Let \( x_0 \in U \). The locally bounded function \( f \) belongs to \( I^\alpha(x_0) \) if

\[
\exists C_{x_0}, r_0(x_0) > 0, \quad \forall r \leq r_0(x_0), \quad \sup_{|x-x_0| \leq r} |f(x) - f(x_0)| \geq C_{x_0} r^\alpha.
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- The bounded function \( f \) belongs to \( I^\alpha(U) \) if
  \[
  \exists C, r_0 > 0, \forall x_0 \in U, \forall r \leq r_0, \sup_{x \in U, |x-x_0| \leq r} |f(x) - f(x_0)| \geq C r^\alpha.
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- In such a case the function $f$ is said to be **strongly monoHölder** of exponent $\alpha$ on $U$.
- The set of strongly monoHölder functions on $U$ is denoted $SM^\alpha(U)$.
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**Two classical examples of strongly monoHölder functions**

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**Two classical examples of strongly monoHölder functions**
- The Weierstrass function $W_H$ is strongly monoHölder of exponent $H$.
- Fractional Brownian Motion $\{B_H(t)\}_{t \in \mathbb{R}}$ is strongly monoHölder of exponent $H$. 
If $f \in I^\alpha(U)$ we can give a lower bound of the box dimension of $\Gamma(f, U)$

$$\dim_B(\Gamma(f, U)) \geq d + 1 - \alpha.$$ 

This inequality is false in general if we replace the box dimension with the Hausdorff dimension.
Nevertheless it is satisfied by most of the studied strongly monoHölder models.
Can we give a lower bound of $\dim_H(\Gamma(f, U))$ for "almost every function" of $C^\alpha(\mathbb{R}^d)$?
Two natural questions

- Can we give a lower bound of \( \dim_H(\Gamma(f, U)) \) for "almost every function" of \( C^\alpha(\mathbb{R}^d) \)?
- Is "almost every function" of \( C^\alpha(\mathbb{R}^d) \) strongly monoHölder of exponent \( \alpha \)?
The concept of prevalence
A natural extension of "almost everywhere" in Banach spaces

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- These properties are a consequence of the $\sigma$–finiteness and translation–invariance of this measure.
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- Unfortunately, there is no such measure in infinite dimensional spaces.
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A suitable concept of "almost everywhere" has to be defined using another approach. In $\mathbb{R}^d$, a Borel set $B$ has Lebesgues measure zero if and only if there exists a compactly supported probability measure $\mu$ such that,

$$\forall x \in \mathbb{R}^d, \quad \mu(x + B) = 0.$$ 

The idea of Christensen (1972)

In a Banach space $E$, a Borel set $B \subset E$ is Haar-null if there exists a compactly supported measure $\mu$ on $E$ such that

$$\forall x \in E, \quad \mu(x + B) = 0.$$
The concept of prevalence
A natural extension of "almost everywhere" in Banach spaces

Definition
A subset $S$ of $E$ is Haar-null if it is included in a Haar-null Borel set.

Definition
The complement of a Haar-null set is called a prevalent set.
Main results

Using the concept of prevalence we can state our two main results

Theorem 1
For any $\alpha \in (0, 1)$, the space $SM^\alpha(\mathbb{R}^d)$ is a prevalent subset of $C^\alpha(\mathbb{R}^d)$.

Theorem 2
For any $\alpha \in (0, 1)$ and for any $f$ in a prevalent subset of $C^\alpha(\mathbb{R}^d)$

$$\dim_H(\Gamma(f, U)) = d + 1 - \alpha.$$
Proof of Theorem 1

Two intermediate results

- A sufficient condition on wavelet coefficients for a function $f$ to be uniformly irregular.
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- A sufficient condition on wavelet coefficients for a function $f$ to be uniformly irregular.
- A general technique for proving prevalent result: the technique of stochastic process.
Proof of Theorem 1
First ingredient of the proof: wavelets

Suppose we are dealing with compactly supported wavelets.
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Notation: wavelet leaders
For any dyadic cube $\lambda$, set

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$$
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Proposition
Let $\alpha \in (0, 1)$. If there exist $C_1, C_2 > 0$ such that for any $\lambda$ of scale $j$,

$$C_12^{-j\alpha} \leq d_\lambda \leq C_22^{-j\alpha},$$

then $f$ is strongly monohölder of exponent $\alpha$. 
Proof of Theorem 1

Second ingredient of the proof: the stochastic process technique

- A random element $X$ on a Banach space $E$ is a measurable mapping $X$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $E$. 

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Proof of Theorem 1
Second ingredient of the proof: the stochastic process technique

- A random element $X$ on a Banach space $E$ is a measurable mapping $X$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $E$.
- For any random element $X$ on $E$, $\mathbb{P}_X(A) = \mathbb{P}\{X \in A\}$ is a probability on $E$. 
Proof of Theorem 1
Second ingredient of the proof: the stochastic process technique

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The stochastic process technique

Set $\mu = \mathbb{P}_X$ in the definition of a Haar-null set. Then to prove that a set $A$ is Haar-null, it is sufficient to find some random element $X$ on $E$ such that

$$\forall f \in E, \quad \mathbb{P}_X(A + f) = 0.$$
Proof of Theorem 1
Prevalent behavior of functions of $C^\alpha(\mathbb{R}^d)$

**Proposition**

For $f$ in a prevalent subset of $C^\alpha(\mathbb{R}^d)$, there exists $C_0 > 0$ such that for any dyadic cube $\lambda$ of scale $j$

$$|d_\lambda| \geq C_0 2^{-j\alpha}.$$
**Proposition**

For \( f \) in a prevalent subset of \( C^\alpha(\mathbb{R}^d) \), there exists \( C_0 > 0 \) such that for any dyadic cube \( \lambda \) of scale \( j \)

\[
|d_\lambda| \geq C_0 2^{-j\alpha}.
\]

**Proof**

Let \((n_{j,k}^{(i)})_{i,j,k}\) be i.i.d. Bernoulli random variables and

\[
X(x) = \sum_{i=1}^{2^d-1} \sum_{j \geq 0} \sum_{|k| \leq 2^{jd}} (-1)^{n_{j,k}^{(i)}} 2^{-\alpha j} \psi_\lambda(x).
\]

We apply the stochastic process technique with \( X \) as random element on \( C^\alpha(\mathbb{R}^d) \).
Proposition (Roueff, 2003)

Let $X$ be the following random wavelet series

$$X(x) = \sum_{\lambda} c_{\lambda} \psi_{\lambda}(x),$$

where $c_{\lambda}$ are independent centered Gaussian random variables with standard deviation $\sigma_{\lambda}$. Define

$$s = \limsup_{J \to \infty} \liminf_{j \to \infty} \left( -j \right)^{-1} \log_2 \min_{j \leq l \leq j + J} \sum_{k} \min(1, \frac{2^{-l}}{\sqrt{2\pi \sigma_{\lambda}}}) 2^{-2l}.$$

Then almost surely $\dim_{\mathcal{H}} \Gamma(X + f, l) \geq s$. 
Proof of Theorem 2

Let \((\xi_{j,k}^{(i)})_{i,j,k}\) be i.i.d. standard Gaussian random variables. We consider the following Gaussian field

\[
X(x) = \sum_{i=1}^{2^d-1} \sum_{j \geq 0} \sum_{|k| \leq 2^j} \frac{\xi_{j,k}^{(i)}}{j^2 \sqrt{\log j}} 2^{-\alpha j} \psi_j^{(i)}(x).
\]

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Bibliography


