How fractal is the sum of two random fractals?

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Papers

- The algebraic difference of two random Cantor sets: the Larsson family (with Károly Simon and Balázs Székely). to appear AoP (2011)
- Correlated fractal percolation and the Palis conjecture (with Henk Don) JSP (2010)
Palis conjecture

Let $F_1$ and $F_2$ be Cantor sets. If

$$\dim_H F_1 + \dim_H F_2 > 1$$

then *generically* it should be true that

$$F_2 - F_1 \text{ contains an interval.}$$

$$F_2 - F_1 := \{y - x : x \in F_1, y \in F_2\}$$
Classical middle third Cantor set

\[ F^0 = [0, 1] \]

\[ F^1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right] \]

\[ F^2 = \left[ 0, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{4}{9} \right] \cup \left[ \frac{8}{9}, 1 \right] \]

\[ F = \bigcap_{n=0}^{\infty} F^n. \]

\[ \dim_H F = \frac{\log 2}{\log 3} \]
A useful observation

Let \( \text{Proj}_{45^\circ}(\cdot) \) be the projection on the \( y \) axis along lines having a \( 45^\circ \) angle with the \( x \)-axis. Then \( F_2 - F_1 = \text{Proj}_{45^\circ}(F_1 \times F_2) \).

\[
\begin{align*}
F_1^1 \times F_2^1 &= y = x - c
\end{align*}
\]
At the next level

\[ F_1^2 \times F_2^2 \]
Slightly smaller intervals, $a < 1/3$
At the next level, $a < 1/3$

\[ F^2 = [0, a^2] \cup [a(1 - a), a] \cup [(1 - a, 1 - a + a^2)] \cup [1 - a^2, 1] \]
Random M-adic Cantor sets

$M \geq 2$: integer

Alphabet: $\mathbb{A} := \{0, \ldots, M - 1\}$.

$\mu$: probability measure on $2^{\mathbb{A}}$ “the joint survival distribution”

$\mathcal{T}$: The $M$-ary tree, i.e., the set of all strings $i_1 \ldots i_n$ over $\mathbb{A}$: nodes.

The nodes $i_1 \ldots i_n$ are labelled by labels $X_{i_1 \ldots i_n}$ from $\{0, 1\}$

$\mathbb{P}_\mu$: probability measure on the space $\{0, 1\}^\mathcal{T}$ of all labelled trees given by

$\mathbb{P}_\mu (X_{\emptyset} = 1) = 1$ and the sets

$$\left\{ i_{n+1} \in \mathbb{A} : X_{i_1 \ldots i_n i_{n+1}} = 1 \right\}$$

are i.i.d $\mu$ for all $i_1 \ldots i_n \in \mathcal{T}$. 
Nodes code $M$-adic intervals

The $n$-th level $M$-adic subintervals of $[0, 1]$:

$$I_{i_1 \ldots i_n} := \left[ \frac{i_1 + \cdots + i_n}{M^n}, \frac{i_1 + \cdots + i_n + i_{n+1}}{M^n} \right] \iff i_1 \ldots i_n,$$

for all $i_1 \ldots i_n \in \mathcal{T}$.

The $n$-th level surviving nodes:

$$S_n := \{i_1 \ldots i_n : X_0 = X_{i_1} = \cdots = X_{i_1 \ldots i_n} = 1\},$$

The random Cantor set $F$:

$$F := \bigcap_{n=0}^{\infty} F^n = \bigcap_{n=0}^{\infty} \bigcup_{i_1 \ldots i_n \in S_n} I_{i_1 \ldots i_n}. $$
‘Marginal’ probabilities

The vector of marginal probabilities
\( \mathbf{p} := (p_0, \ldots, p_{M-1}) : \quad p_i := \mathbb{P}_\mu (X_i = 1) , \quad \text{for all } i \in \mathbb{A} . \)

The traditional deterministic triadic (so \( M = 3 \)) Cantor set is obtained with the measure \( \mu \) defined by \( \mu(\{0, 2\}) = 1 \), it has vector of marginal probabilities \( \mathbf{p} = (1, 0, 1) \).

EXAMPLE: \( \mu(\{0\}) = 1/2, \; \mu(\{0, 2\}) = 1/2 \Rightarrow \mathbf{p} = (1, 0, \frac{1}{2}) \)

The first three levels of a realization of the labeled tree \( (X_{i_1 \ldots i_n}) \) interspersed with the surviving intervals in the approximations \( F^n \):
EXAMPLE: Mandelbrot percolation
also called fractal percolation

Given: a parameter $p$ with $0 \leq p \leq 1$.

$$\mu(B) = p^{\#B}(1 - p)^{M - \#B} \quad \text{for} \quad B \subseteq \mathbb{A}.$$ 

Here the marginal probabilities are

$$p = (p, p, \ldots, p).$$

This is the case where the subintervals are chosen independently and with the same probability.
When is $F$ non-empty?

Branching process: $(\#S_n)$ with offspring: the distribution of $\#S_1$. $F = \emptyset$ if and only if the branching process $(\#S_n)$ dies out.

We have

$$\mathbb{E}_\mu \#S_1 = p_0 + \cdots + p_{M-1} = \|p\|_1.$$  

Thus $F \neq \emptyset$ with positive probability if and only if

$$\|p\|_1 > 1 \quad \text{or} \quad \mathbb{P}_\mu(\#S_1 = 1) = 1.$$
Hausdorff dimension of $F$

**THEOREM** (Falconer, or Mauldin & Williams, 1986). The Hausdorff dimension of $F$ is equal to

$$\dim_H F = \frac{\log (\mathbb{E}_\mu \# S_1)}{\log (M)} = \frac{\log (\|p\|_1)}{\log (M)}$$

with probability one.
When does $F_2 - F_1$ contain an interval?

Define $p_{M+j} = p_j$ for $j = 0, 1, \ldots, M-1$.
With this we define the *cyclic autocorrelations* $\gamma_k$ by

$$
\gamma_k = \sum_{j=0}^{M-1} p_j p_{j+k} \quad \text{for } k = 0, \ldots, M - 1.
$$

**THEOREM**

Conditional on $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$

(a) If $\gamma_k > 1$ for all $k$ then $F_2 - F_1$ contains an interval almost surely.

(b) If there exists a $k \in \{0, \ldots, M - 1\}$ such that $\gamma_k < 1$ and $\gamma_{k+1} < 1$ then $F_2 - F_1$ almost surely does not contain an interval.
For $M = 2$ the Theorem tells you nothing about the case \[ \gamma_0 > 1, \gamma_1 < 1. \]

For $M = 3$ our Theorem covers all possibilities:

\[ \gamma_0 = p_0^2 + p_1^2 + p_2^2, \]
\[ \gamma_1 = p_0p_1 + p_1p_2 + p_0p_2, \]
\[ \gamma_2 = p_0p_2 + p_1p_0 + p_2p_1. \]

So \[ \gamma_0 \geq \gamma_1 = \gamma_2. \]
A useful observation, part 2

Remember: \( F_2 - F_1 = \text{Proj}_{45^\circ} (F_1 \times F_2) \).

Since it is easier to study the \( 90^\circ \) projection we rotate the \([0,1] \times [0,1]\) square by \( 45^\circ \) in the positive direction and translate it, so that its horizontal diagonal is the \( x \) axis.

The \( n \)-th level (rotated) squares: \( Q_{i_1...i_n,j_1...j_n} \).

Any \( Q_{i_1...i_n,j_1...j_n} \) is divided into two triangles: \( R_{i_1...i_n,j_1...j_n} \) and \( L_{i_1...i_n,j_1...j_n} \).

The orthogonal projection of these Left and Right \( n \)-th level triangles are at most \( 2 \cdot M^n \) intervals.

The proof of the \textsc{Theorem} is based on an analysis of how the numbers of the \( R \) and \( L \) triangles grow (\textit{jointly}) in the vertical columns that project on these intervals.
EXAMPLE with $M = 3$. 

\[
\begin{array}{ccccccccc}
C^L_0 & C^L_1 & C^L_2 & C^R_0 & C^R_1 & C^R_2 \\
C^L_{00} & C^L_{01} & C^L_{02} & C^R_{00} & C^R_{01} & C^R_{02} \\
C^L_{10} & C^L_{11} & C^L_{12} & C^R_{10} & C^R_{11} & C^R_{12} \\
C^L_{20} & C^L_{21} & C^L_{22} & C^R_{20} & C^R_{21} & C^R_{22} \\
\end{array}
\]
Counting triangles

Let $Z^{RR}(k_1 \ldots k_n)$ be the number of $n^{th}$ order $R$-triangles in column $C^R_{k_1 \ldots k_n}$ on the right side of square $Q$.

Similarly, $Z^{RL}(k_1 \ldots k_n)$, $Z^{LR}(k_1 \ldots k_n)$ and $Z^{LL}(k_1 \ldots k_n)$.

This is a 2-type ($R$ and $L$) branching process in a varying environment with neighbour interaction.

No theory!

TRICK: count $\Delta$-pairs, i.e., disjoint pairs of disjoint $R-$ and $L$-triangles.
Expectation matrices

Let $\mathcal{M}(k_1 \ldots k_m) :=$

$$
\begin{bmatrix}
\mathbb{E}Z^{RR}(k_1 \ldots k_m) & \mathbb{E}Z^{RL}(k_1 \ldots k_m) \\
\mathbb{E}Z^{LR}(k_1 \ldots k_m) & \mathbb{E}Z^{LL}(k_1 \ldots k_m)
\end{bmatrix}.
$$

Then from the definition one can easily check that

$$
\mathcal{M}(k_1 \ldots k_m) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_m).
$$

Define $Z^R(k) = Z^{RR}(k) + Z^{LR}(k)$, and similarly $Z^L(k)$.
This is the number of $R$-(respectively $L$)-triangles generated by a $\Delta$-pair in column $C_k$.
By a geometric observation we obtain that

$$
\mathbb{E}Z^L(k) = \gamma_{k+1}, \quad \mathbb{E}Z^R(k) = \gamma_k, \text{ i.e. } [\gamma_k, \gamma_{k+1}] = [1, 1] \mathcal{M}(k).
$$

This is the reason that the numbers $\gamma_k$ play an important role.
Higher order Cantor sets

Idea: ‘collapsing’ $n$ steps of the construction into one step. This gives a random $M^n$-adic Cantor set with joint survival distribution denoted $\mu^{(n)}$.

Alphabet: $A^{(n)} = \{0, \ldots, M^n - 1\}$.

$\mu^{(n)}$ is determined by requiring

$$X_k^{(n)} \sim \prod_{i=1}^{n} X_{k_1 \ldots k_i} = X_{k_1} X_{k_1 k_2} \cdots X_{k_1 \ldots k_n},$$

Higher order marginal probabilities:

$$p_k^{(n)} := \mathbb{P}_{\mu^{(n)}} \left( X_k^{(n)} = 1 \right) = \prod_{i=1}^{n} \mathbb{P}_{\mu} \left( X_{k_1 \ldots k_i} = 1 \right) = \prod_{i=1}^{n} p_{k_i},$$

for all $k = \sum_{i=0}^{n} k_i M^{n-i}$. 
Higher order correlation coefficients

- $\gamma^{(n)}_k > 1$ for all $k \in A^{(n)}$, then we are in the ‘intervals’ case, whereas when
- $\gamma^{(n)}_k, \gamma^{(n)}_{k+1} < 1$ for some $k \in A^{(n)}$, then we are in the ‘no intervals’ case.

Can show that

For all $k_1 \ldots k_n \in T$ and $k = \sum_{i=0}^{n} k_i M^{n-i}$:

$$
\begin{bmatrix}
\gamma^{(n)}_{k+1} & \gamma^{(n)}_k
\end{bmatrix}
= [1, 1] \mathcal{M}^{(n)} (k) = [1, 1] \mathcal{M} (k_1) \cdots \mathcal{M} (k_n).
$$
Classifying $M = 2$

Have to find:

$$\min\{\gamma_k^{(n)} : k \in A^{(n)}\}.$$

This will be very hard for general $M$, but can be done for $M = 2$.

**PROPOSITION** Let $F_1$ and $F_2$ be two independent identically distributed 2-adic random Cantor sets where $\mathbf{p} = (p_0, p_1)$.
If $C > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
If $C < 1$, then $F_1 - F_2$ contains no interval a.s.
Here $C$ is defined by

$$C := p_0p_1(1 + p_0^2 + p_1^2).$$
The \((p_0, p_1)\)-plane

\[
\gamma_0 = 1 \\
\gamma_1 = 1 \\
C = 1 \\
dim_H(F) = \frac{1}{2}
\]
The lower spectral radius

\[ \| \cdot \|: \text{ a submultiplicative norm on } \mathbb{R}^{d \times d} \]
\[ \Sigma \subset \mathbb{R}^{d \times d}: \text{ finite set of matrices.} \]

Let

\[ \rho_n(\Sigma, \| \cdot \|) := \min_{A_1, \ldots, A_n \in \Sigma} \| A_1 \cdots A_n \|^{1/n}. \]

The lower spectral radius of \( \Sigma \) is

\[ \underline{\rho}(\Sigma) := \liminf_{n \to \infty} \rho_n(\Sigma, \| \cdot \|). \]

EXAMPLE: \( \Sigma = \{ A \} \) then \( \underline{\rho}(\Sigma) = \rho(A) \).
The distributed growth condition

More general: two different survival distributions $\mu$ and $\lambda$.

For $X, Y \subseteq A, e \in A$:

$\gamma_e(X, Y)$ is the $e^{th}$ correlation coefficient from the distributions $\mu^*$ and $\lambda^*$ assigning probability one to $X$ and $Y$ respectively, i.e.,

$$\gamma_e(X, Y) = \sum_{i \in A} 1_Y(i)1_X(i + e).$$

The pair of joint survival distributions $(\mu, \lambda)$ satisfies the distributed growth condition if for all $k \in A$ exists $X_k, Y_k \subseteq A$ with

(DG0) $\mu(X_k) > 0$ and $\lambda(Y_k) > 0$,

(DG1) $\min_{e \in A} \gamma_e(X_k, Y_k) \geq 1$,

(DG2) $\gamma_k(X_k, Y_k) \geq 2, \gamma_{k+1}(X_k, Y_k) \geq 2$.

PROPERTY: $\mu$ satisfies the DGC $\Rightarrow \mu^{(n)}$ satisfies the DGC.
A complete classification when $\mu = \lambda$.

**THEOREM**

$F_1 - F_2$: $M$-adic independent random Cantor sets $F_1$ and $F_2$ whose joint survival distributions satisfy the distributed growth condition, and have equal vectors of marginal probabilities $p = q$. Let $\Sigma_M = \{ M(0), \cdots M(M-1) \}$.

(a) If $\rho(\Sigma_M) > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{ F_1 - F_2 \neq \emptyset \}$.

(b) If $\rho(\Sigma_M) < 1$, then $F_1 - F_2$ contains no intervals a.s.

Tsitsiklis & Blondel (1997): NP-hard to approximate $\rho(\Sigma_M)$ for non-negative matrices.
Larsson’s random Cantor set

Given: two positive numbers $a$ and $b$ such that

$$a > \frac{1}{4} \quad \text{and} \quad 3a + 2b < 1.$$  

$$\dim_H C_{a,b} = -\frac{\log 2}{\log a} \quad \text{so} \quad a > \frac{1}{4} \iff \dim_H C_{a,b} > 1/2,$$

which is equivalent to the Palis condition.
THEOREM Let $C_1$, $C_2$ be independent random Cantor sets having the same distribution as $C_{a,b}$. Then the algebraic difference $C_2 - C_1$ almost surely contains an interval.
Open problems

- What is the Hausdorff dimension of $F_1 - F_2$ when there is no interval?
- Does the spectral radius characterization work if $\mu \neq \lambda$?
- Critical case when $\mathbf{p} \neq (p, p, \ldots, p)$? (equal $p_i$’s solved: no interval)
- Generalizations of Larsson’s random Cantor set?
- What are the essential ingredients (as e.g. symmetry) that make Palis’ conjecture hold?