How fractal is the sum of two random fractals?

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Papers

- On the size of the algebraic difference of two random Cantor sets (with K.Simon). RSA (2008).
- Differences of Random Cantor Sets and the Lower Spectral Radius (with Bram Kuijvenhoven). to appear JEMS (2010).
- The algebraic difference of two random Cantor sets: the Larsson family (with Károly Simon and Balázs Székely). to appear AoP (2011)
- Correlated fractal percolation and the Palis conjecture (with Henk Don) JSP (2010)

Palis conjecture

Let F_1 and F_2 be Cantor sets. If

$$\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1$$

then *generically* it should be true that

$$F_2 - F_1 \text{ contains an interval.}$$

$$F_2 - F_1 := \{y - x : \; x \in F_1, \; y \in F_2\}$$

Classical middle third Cantor set

$$F^0 = [0, 1]$$

$$F^1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

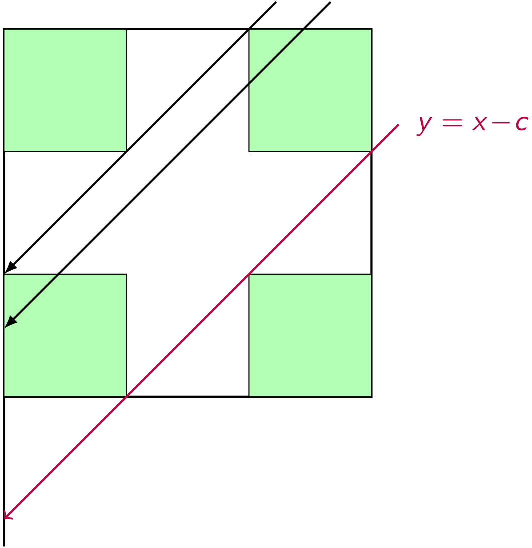
$$F^2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{4}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

$$F = \bigcap_{n=0}^{\infty} F^n.$$

$$\dim_{\mathrm{H}} F = \frac{\log 2}{\log 3}$$

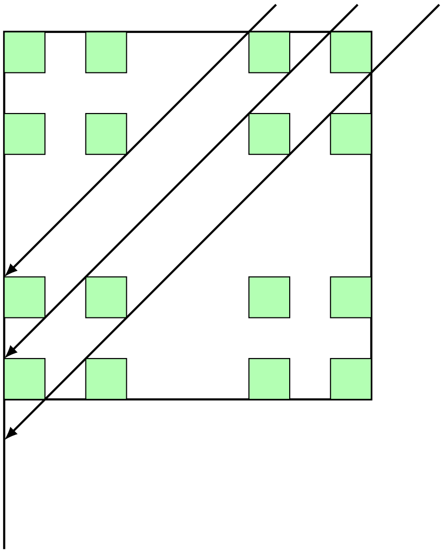
A useful observation

Let $\text{Proj}_{45^\circ}(\cdot)$ be the projection on the y axis along lines having a 45° angle with the x-axis. Then $F_2 - F_1 = \text{Proj}_{45^\circ}(F_1 \times F_2)$.



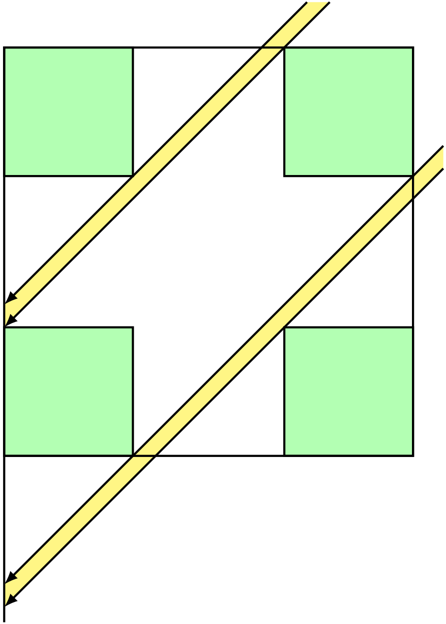
$F_1^1 \times F_2^1$

At the next level

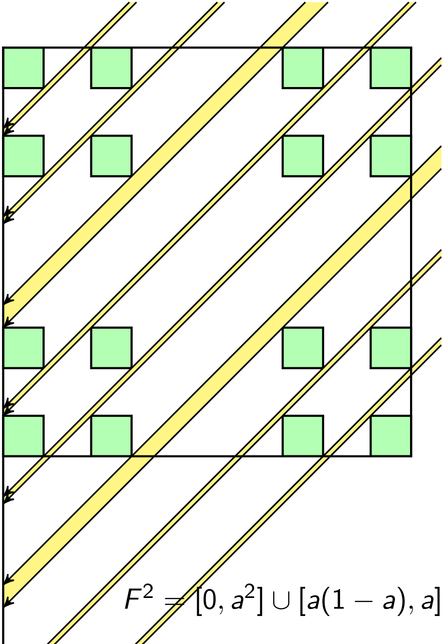


$F_1^{2} \times F_2^{2}$

Slightly smaller intervals, a $< 1/3$



At the next level, a $< 1/3$



$$F^2 = [0, a^2] \cup [a(1-a), a] \cup [(1-a, 1-a+a^2] \cup [1-a^2, 1]$$

Random M-adic Cantor sets

$M \geq 2$: integer

Alphabet: $\mathbb{A} := \{0, \ldots, M-1\}$.

μ: probability measure on $2^{2^{\mathbb{A}}}$ "the *joint survival distribution*"

$\mathcal{T}$: The *M-ary tree*, i.e., the set of all strings $i_1 \ldots i_n$ over $\mathbb{A}$: *nodes*.

The nodes $i_1 \ldots i_n$ are labelled by labels $X_{i_1 \ldots i_n}$ from $\{0, 1\}$

$\mathbb{P}_\mu$: probability measure on the space $\{0, 1\}^{\mathcal{T}}$ of all labelled trees given by
$\mathbb{P}_\mu (X_\emptyset = 1) = 1$ and the sets

$$\left\{ i_{n+1} \in \mathbb{A} : X_{i_1 \ldots i_n i_{n+1}} = 1 \right\}$$

are i.i.d μ for all $i_1 \ldots i_n \in \mathcal{T}$.

Nodes code M-adic intervals

The n-th level M-adic subintervals of $[0,1]$:

$$I_{i_1\ldots i_n} := \left[\frac{i_1+\cdots+i_n}{M^n}, \frac{i_1+\cdots+i_n+i_{n+1}}{M^n}\right] \quad \Leftrightarrow \quad i_1\ldots i_n,$$

for all $i_1\ldots i_n \in \mathcal{T}$.

The n-th level surviving nodes:

$$S_n := \{i_1\ldots i_n : X_\emptyset = X_{i_1} = \cdots = X_{i_1\ldots i_n} = 1\},$$

The random Cantor set F:

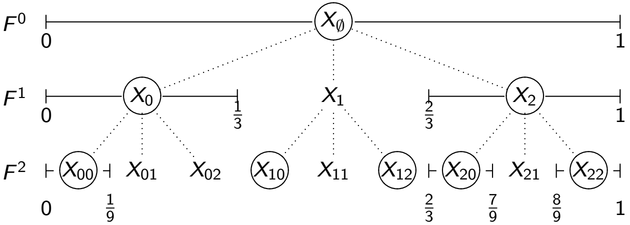
$$F := \bigcap_{n=0}^{\infty} F^n = \bigcap_{n=0}^{\infty} \bigcup_{i_1\ldots i_n \in S_n} I_{i_1\ldots i_n}.$$

‘Marginal’ probabilities

The vector of marginal probabilities
$\mathbf{p} := (p_0, \ldots, p_{M-1}) : \qquad p_i := \mathbb{P}_\mu (X_i = 1), \qquad$ for all $i \in \mathbb{A}$.

The traditional deterministic triadic (so $M = 3$) Cantor set is obtained with the measure μ defined by $\mu(\{0,2\}) = 1$, it has vector of marginal probabilities $\mathbf{p} = (1, 0, 1)$.

EXAMPLE: $\mu(\{0\}) = 1/2,\ \mu(\{0,2\}) = 1/2 \Rightarrow \mathbf{p} = (1, 0, \frac{1}{2})$
The first three levels of a realization of the labeled tree $(X_{i_1 \ldots i_n})$ interspersed with the surviving intervals in the approximations F^n:



EXAMPLE: Mandelbrot percolation

also called fractal percolation

Given: a parameter p with $0 \leq p \leq 1$.

$$\mu(B) = p^{\#B}(1-p)^{M-\#B} \quad \text{for} \quad B \subseteq \mathbb{A}.$$

Here the marginal probabilities are

$$\mathbf{p} = (p, p, \ldots, p).$$

This is the case where the subintervals are chosen independently and with the same probability.

When is F non-empty?

Branching process: $(\#S_n)$ with offspring: the distribution of $\#S_1$.
$F = \emptyset$ if and only if the branching process $(\#S_n)$ dies out.

We have

$$\mathbb{E}_\mu \#S_1 = p_0 + \cdots + p_{M-1} = \|\mathbf{p}\|_1.$$

Thus $F \neq \emptyset$ with positive probability if and only if

$$\|\mathbf{p}\|_1 > 1 \quad \text{or} \quad \mathbb{P}_\mu(\#S_1 = 1) = 1.$$

Hausdorff dimension of F

THEOREM (Falconer, or Mauldin & Williams, 1986).
The Hausdorff dimension of F is equal to

$$\dim_{\mathrm{H}} F = \frac{\log\left(\mathbb{E}_\mu \# S_1\right)}{\log(M)} = \frac{\log\left(\|\mathbf{p}\|_1\right)}{\log(M)}$$

with probability one.

When does $F_2 - F_1$ contain an interval?

Define $p_{M+j} = p_j$ for $j = 0,1,\ldots,M-1$.
With this we define the *cyclic autocorrelations* γ_k by

$$\gamma_k = \sum_{j=0}^{M-1} p_j p_{j+k} \quad \text{for } k = 0,\ldots,M-1.$$

THEOREM

Conditional on $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$

(a) If $\gamma_k > 1$ for all k then $F_2 - F_1$ contains an interval almost surely.

(b) If there exists a $k \in \{0,\ldots,M-1\}$ such that $\gamma_k < 1$ and $\gamma_{k+1} < 1$ then $F_2 - F_1$ almost surely does not contain an interval.

REMARKS

- For $M = 2$ the Theorem tells you nothing about the case

$$\gamma_0 > 1, \gamma_1 < 1.$$

- For $M = 3$ our Theorem covers all possibilities:

$$\gamma_0 = p_0^2 + p_1^2 + p_2^2,$$

$$\gamma_1 = p_0 p_1 + p_1 p_2 + p_0 p_2,$$

$$\gamma_2 = p_0 p_2 + p_1 p_0 + p_2 p_1.$$

So

$$\gamma_0 \geq \gamma_1 = \gamma_2.$$

A useful observation, part 2

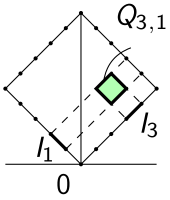
Remember: $F_2 - F_1 = \text{Proj}_{45^\circ}(F_1 \times F_2)$.

Since it is easier to study the 90° projection we rotate the $[0,1] \times [0,1]$ square by 45° in the positive direction and translate it, so that its horizontal diagonal is the x axis.

The n-th level (rotated) squares: $Q_{i_1 \ldots i_n, j_1 \ldots j_n}$.

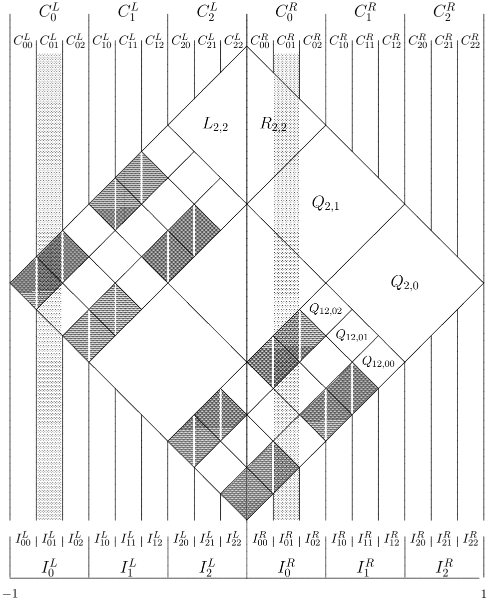
Any $Q_{i_1 \ldots i_n, j_1 \ldots j_n}$ is divided into two triangles: $R_{i_1 \ldots i_n, j_1 \ldots j_n}$ and $L_{i_1 \ldots i_n, j_1 \ldots j_n}$.

The orthogonal projection of these Left and Right n-th level triangles are at most $2 \cdot M^n$ intervals.



The proof of the THEOREM is based on an analysis of how the numbers of the R and L triangles grow (jointly) in the vertical columns that project on these intervals.

EXAMPLE with $\mathbf{M = 3}$.



Counting triangles

Let $Z^{RR}(k_1 \dots k_n)$ be the number of n^{th} order R-triangles in column $C^R_{k_1 \dots k_n}$ on the right side of square Q.

Similarly, $Z^{RL}(k_1 \dots k_n)$, $Z^{LR}(k_1 \dots k_n)$ and $Z^{LL}(k_1 \dots k_n)$.

This is a 2-type (R and L) branching process in a varying environment with neighbour interaction.

No theory!

TRICK: count Δ-pairs, i.e., disjoint pairs of disjoint $R-$ and L-triangles.

Expectation matrices

Let $\mathcal{M}(k_1 \dots k_m) :=$

$$\begin{bmatrix} \mathbb{E}Z^{RR}(k_1 \dots k_m) & \mathbb{E}Z^{RL}(k_1 \dots k_m) \\ \mathbb{E}Z^{LR}(k_1 \dots k_m) & \mathbb{E}Z^{LL}(k_1 \dots k_m) \end{bmatrix}.$$

Then from the definition one can easily check that

$$\mathcal{M}(k_1 \dots k_m) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_m).$$

Define $Z^R(k) = Z^{RR}(k) + Z^{LR}(k)$, and similarly $Z^L(k)$.
This is the number of R-(respectively L)-triangles generated by a Δ-pair in column C_k.
By a geometric observation we obtain that

$$\mathbb{E}Z^L(k) = \gamma_{k+1}, \ \ \mathbb{E}Z^R(k) = \gamma_k, \ \text{i.e.} \ [\gamma_k, \gamma_{k+1}] = [1, \ 1]\mathcal{M}(k).$$

This is the reason that the numbers γ_k play an important role.

Higher order Cantor sets

Idea: ‘collapsing’ n steps of the construction into one step. This gives a random M^n-adic Cantor set with joint survival distribution denoted $\mu^{(n)}$.
Alphabet: $\mathbb{A}^{(n)} = \{0, \ldots, M^n - 1\}$.
$\mu^{(n)}$ is determined by requiring

$$X_k^{(n)} \sim \prod_{i=1}^{n} X_{k_1 \ldots k_i} = X_{k_1} X_{k_1 k_2} \cdots X_{k_1 \ldots k_n},$$

Higher order marginal probabilities:

$$p_k^{(n)} := \mathbb{P}_{\mu^{(n)}} \left(X_k^{(n)} = 1 \right) = \prod_{i=1}^{n} \mathbb{P}_{\mu} \left(X_{k_1 \ldots k_i} = 1 \right) = \prod_{i=1}^{n} p_{k_i},$$

for all $k = \sum_{i=0}^{n} k_i M^{n-i}$.

Higher order correlation coefficients

- $\gamma_k^{(n)} > 1$ for all $k \in \mathbb{A}^{(n)}$, then we are in the 'intervals' case, whereas when
- $\gamma_k^{(n)}, \gamma_{k+1}^{(n)} < 1$ for some $k \in \mathbb{A}^{(n)}$, then we are in the 'no intervals' case.

Can show that

For all $k_1 \dots k_n \in \mathcal{T}$ and $k = \sum_{i=0}^{n} k_i M^{n-i}$:

$$\left[\gamma_{k+1}^{(n)} \quad \gamma_k^{(n)}\right] = [1,\ 1]\, \mathcal{M}^{(n)}(k) = [1,\ 1]\, \mathcal{M}(k_1) \cdots \mathcal{M}(k_n)\,.$$

Classifying $\mathbf{M} = 2$

Have to find:

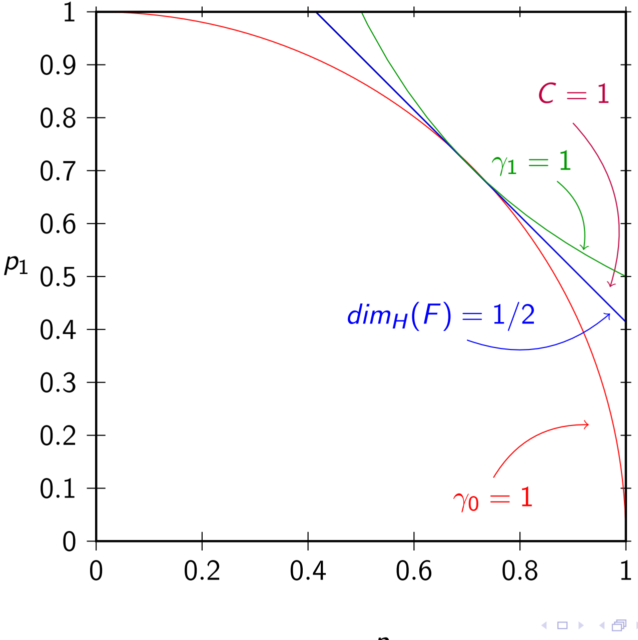
$$\min\{\gamma_k^{(n)} : k \in \mathbb{A}^{(n)}\}.$$

This will be very hard for general M, but can be done for $M = 2$.

PROPOSITION Let F_1 and F_2 be two independent identically distributed 2-adic random Cantor sets where $\mathbf{p} = (p_0, p_1)$.
If $C > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
If $C < 1$, then $F_1 - F_2$ contains no interval a.s.
Here C is defined by

$$C := p_0 p_1 (1 + p_0^2 + p_1^2).$$

The $(\mathbf{p_0}, \mathbf{p_1})$-plane



The lower spectral radius

$\|\cdot\|$: a submultiplicative norm on $\mathbb{R}^{d\times d}$
$\Sigma \subset \mathbb{R}^{d\times d}$: finite set of matrices.
Let

$$\underline{\rho}_n(\Sigma, \|\cdot\|) := \min_{A_1,\dots,A_n\in\Sigma} \|A_1\cdots A_n\|^{1/n}.$$

The *lower* spectral radius of Σ is

$$\underline{\rho}(\Sigma) := \liminf_{n\to\infty} \underline{\rho}_n(\Sigma, \|\cdot\|).$$

EXAMPLE: $\Sigma = \{A\}$ then $\underline{\rho}(\Sigma) = \rho(A)$.

The distributed growth condition

More general: two different survival distributions μ and λ.

For $X, Y \subseteq \mathbb{A}$, $e \in \mathbb{A}$:
$\gamma_e(X, Y)$ is the e^{th} correlation coefficient from the distributions $\mu^\star$ and $\lambda^\star$ assigning probability one to X and Y respectively, i.e.,

$$\gamma_e(X, Y) = \sum_{i \in \mathbb{A}} \mathbf{1}_Y(i)\mathbf{1}_X(i+e). \tag{1}$$

The pair of joint survival distributions (μ, λ) satisfies the *distributed growth condition* if for all $k \in \mathbb{A}$ $\exists$ $X_k, Y_k \subseteq \mathbb{A}$ with

(DG0) $\mu(X_k) > 0$ and $\lambda(Y_k) > 0$,

(DG1) $\min_{e \in \mathbb{A}} \gamma_e(X_k, Y_k) \geq 1$,

(DG2) $\gamma_k(X_k, Y_k) \geq 2,\ \gamma_{k+1}(X_k, Y_k) \geq 2$.

PROPERTY: μ satisfies the DGC $\Rightarrow$ $\mu^{(n)}$ satisfies the DGC.

A complete classification

when $\mu = \lambda$.

THEOREM

$F_1 - F_2$: M-adic independent random Cantor sets F_1 and F_2 whose joint survival distributions satisfy the distributed growth condition, and have equal vectors of marginal probabilities $\mathbf{p} = \mathbf{q}$. Let $\Sigma_{\mathcal{M}} = \{\mathcal{M}(0), \cdots \mathcal{M}(M-1)\}$.

(a) If $\underline{\rho}(\Sigma_{\mathcal{M}}) > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.

(b) If $\underline{\rho}(\Sigma_{\mathcal{M}}) < 1$, then $F_1 - F_2$ contains no intervals a.s.

Tsitsiklis & Blondel (1997): NP-hard to approximate $\underline{\rho}(\Sigma_{\mathcal{M}})$ for non-negative matrices.

Larsson's random Cantor set

Given: two positive numbers a and b such that

$$a > \frac{1}{4} \quad \text{and} \quad 3a + 2b < 1.$$

$$\dim_{\mathrm{H}} C_{a,b} = -\frac{\log 2}{\log a} \quad \textit{so} \quad a > \frac{1}{4} \Leftrightarrow \dim_{\mathrm{H}} C_{a,b} > 1/2,$$

which is equivalent to the Palis condition.

Larsson's random Cantor set 2

THEOREM Let C_1, C_2 be independent random Cantor sets having the same distribution as $C_{a,b}$. Then the algebraic difference $C_2 - C_1$ almost surely contains an interval.

Open problems

- What is the Hausdorff dimension of $F_1 - F_2$ when there is no interval?
- Does the spectral radius characterization work if $\mu \neq \lambda$?
- Critical case when $\mathbf{p} \neq (p, p, \ldots, p)$?
  (equal p_i's solved: no interval)
- Generalizations of Larsson's random Cantor set?
- What are the essential ingredients (as e.g. symmetry) that make Palis' conjecture hold?