Testing the type of a semimartingale: Itō vs. Multifractal

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Motivation

- This work draws heavy inspiration from a series of papers in mathematical finance by Aït-Sahalia and Jacod (2008, 2009) that base themselves on the behaviour of structure functions of Itō semimartingales to build nonparametric test statistics concerning the type of the Itō semimartingale.
- We wanted to broaden their results to a class of martingales (the MRW's of Bacry and Muwy (2002)) that are not Itō, since indeed the behaviour of the structure functions of MRW's or more generally random cascades is very specific.

Two classes of models for financial prices

Mathematical results

A simulation study

“Mainstream” model for financial prices

Common assumption in financial mathematics: prices are martingales (unpredictible processes, zero-sum game) or more generally semimartingales. In practice, most models are members of the class of Itō semimartingales, ie. sum of continuous Brownian integral and a jump process:

$$X_t = \int_0^t \sigma_s dB_s + J_t,$$

where the process σ can itself be random and depend on the Brownian motion B and the jump process J. This gives a very large class of models, which can take into account a wide range of statistical regularities observed in practice.

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Statistical tests for the type of an Itō semimartingales

We may then want to decide which type of Itō semimartingale is in adequation with our data. Is the jumps part zero? is the Brownian part zero?

Aït Sahalia and Jacod (2008, 2009) construct nonparametric tests for answering these questions that are based on the behaviour of structure functions

$$S_N(p) = \sum_{k=0}^{N-1} |X_{(k+1)/N} - X_{k/N}|^p$$

- if X has no jumps, $S_N(p) \sim N^{1-p/2}$ for $p \geq 0$
- if X has jumps, $S_N(p) \sim N^{\max(0,1-p/2)}$ for $p \geq 0$.

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Multifractal models for financial prices

- Mandelbrot (1997): prices behave as $X_t = B_{\theta_t}$, where θ is a positive, continuous, increasing "cascade" process independent of the Brownian motion B and such that $\mathbb{E}[(\theta_{t+s} - \theta_t)^p] = \mathbb{E}[|X_{t+s} - X_t|^{2p}] \approx c_{2p} s^{\zeta_{2p}}$ as $s \to 0$. In contrast with Itō semimartingales, $p \mapsto \zeta_p$ is a strictly concave function with $\zeta_0 = \zeta_2 = 0$.
- Construction of a dyadic cascade: Take some iid positive random variable W_i, $i \in \{0,1\}^n$, $n \in \mathbb{N}$ such that $\mathbb{E}[W_i] = 1$. Define for $0 \le t \le 1$

$$\theta_t^1 = \int_0^t \left(W_0 1_{u \in [0,1/2]} + W_1 1_{u \in (1/2,1]} \right) du$$

$$\begin{aligned}\theta_t^2 = \int_0^t (& W_0 W_{00} 1_{u \in [0,1/4]} + W_0 W_{01} 1_{u \in (1/4,1/2]} \\ + & W_1 W_{10} 1_{u \in (1/2,3/4]} + W_1 W_{11} 1_{u \in (3/4,1]}) du \\ \ldots & \end{aligned}$$

Then $\theta^n \to \theta$ as $n \to +\infty$ (Kahane and Peyrière 1976).

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Multifractal martingales

- Calvet and Fisher (2001), Barral and Mandelbrot (2002), Bacry and Muzy (2002): grid-free, stationary construction of a cascade process. Idea: replace the iterated multiplication of iid positive random variables with the exponential of a Lévy process.
- The MRW of Bacry and Muzy has the nice scaling property: for $r \in [0, 1]$

$$(X_{rt},\, 0 \le t \le L) \stackrel{d}{=} r^{1/2} e^{W_r} (X_t,\, 0 \le t \le L),$$

  where W_r is an infinitely divisible random variable independent of X.
- All classes of cascades θ have the property of being increasing processes that are not absolutely continuous wrt the Lebesgue measure. It follows that $X_t = B_{\theta_t}$ cannot be written as a Brownian integral $\int_0^t \sigma_s dB_s$: X is a continuous martingale that is not Itō.

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General approach

Our aim is to complete the results of Aït-Sahalia and Jacod by taking into account the possibility that X may be an MRW.

- Abry, Jaffard, Roux and Wendt (2007, 2008) give a long and numerically involved study of how to estimate the scaling exponent ζ_p in practice. One should use *wavelet leaders* and *log-cumulants* rather than increments and structure functions.
- We however manage to prove a CLT for the structure function of an MRW, which would be difficult to achieve for log-cumulants and/or wavelet leaders.

This enables us to propose consistent nonparametric tests for H_0: "X is Itō" against H_1: "X is an MRW" and conversely. This is not testing multifractality: there are some Itō semimartingales with nondegenerate singularity spectrum (Jaffard, 1997).

Case H_0: X is Itō

Proposition 1 (Aït-Sahalia and Jacod)

If X is Itō with no jumps, then

$$\sqrt{\frac{7}{32}}\frac{S_N(4)}{(S_N(8))^{1/2}}\left(\frac{S_{N/2}(4)}{S_N(4)}-2\right)\xrightarrow{\mathcal{L}} N(0,1).$$

From the strict concavity of $p \mapsto \zeta_p$, the same quantity goes to $+\infty$ if X is an MRW. However, it goes to an unobservable random variable if X is Itō with jumps.

Test for H_0: X is Itō

Let (k_N) be a sequence such that $k_N \leq 1$, $k_N \to 1$ and $(1 - k_N)\log(N) \to +\infty$.

Theorem 1

Define

$$T_N^{Ito} = \sqrt{\frac{7}{32}} N^{k_N - 1} \frac{S_{\lfloor N^{k_N} \rfloor}(4)}{(S_N(8))^{1/2}} \left(\frac{S_{N/2}(4)}{S_N(4)} - 2\right).$$

Then if X is Itō with jumps, $T_N^{Ito} \to 0$ in probability. If X is Itō with no jumps, $T_N^{Ito} \xrightarrow{\mathcal{L}} N(0,1)$. If X is an MRW, $T_N^{Ito} \to +\infty$ in probability.

Case H_0: X is an MRW

Proposition 2

If X is an MRW, then

$$\frac{\sqrt{3}}{\sqrt{2(2^{\zeta_4-1})}}\frac{S_N(2)-S_{N/2}(2)}{\sqrt{S_N(4)}}\xrightarrow{\mathcal{L}} N(0,1).$$

If X is Itō with jumps, this quantity goes to 0. However, if X is Itō with no jumps, this quantity is of order 1.

Remark: Ossiander and Waymire (2000) and Ludeña (2008) also prove CLT's for random cascades, but these CLT's comport unobserved centering terms, which makes them hard to use for building statistical tests.

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Test for the case H_0: X is an MRW

Theorem 2

Choose $c \in (0,1)$ and define

$$T_N^{MRW} = \frac{\sqrt{3}}{\sqrt{2(2^{\zeta_4 - 1})}} N^{(1-c)\zeta_4/2} \frac{S_N(2) - S_{N/2}(2)}{\sqrt{S_{\lfloor N^c \rfloor}(4)}}$$

Then if X is Itō, T_N^{MRW} goes to 0 in probability. If X is an MRW, then $T_N^{MRW} \xrightarrow{\mathcal{L}} N(0,1)$.

Sketch of a proof for Proposition 2

1. $N^{-1+\zeta_p} S_N(p) \to \theta^{(p)}$ as $N \to +\infty$ (Ludeña 2008, D. 2009). In the case $p = 2$, $\theta^{(2)}$ is simply the same as θ.
2. Define

$$\xi_k^{N,1} = N^{\zeta_4/2}\big((X_{2k/N} - X_{(2k-1)/N})^2 + (X_{(2k-1)/N} - X_{(2k-2)/N})^2 - (\theta_{2k/N} - \theta_{(2k-2)/N})\big)$$

and

$$\xi_k^{N,2} = N^{\zeta_4/2}\big((X_{2k/N} - X_{(2k-2)/N})^2 - (\theta_{2k/N} - \theta_{(2k-2)/N})\big).$$

Then $(\sum_{k=1}^n \xi_k^{N,1}, \sum_{k=1}^n \xi_k^{N,2})_n$ is a 2d martingale.

3. Check that the Lindeberg condition holds and apply a martingale CLT (Jacod and Shiryaev 2002).

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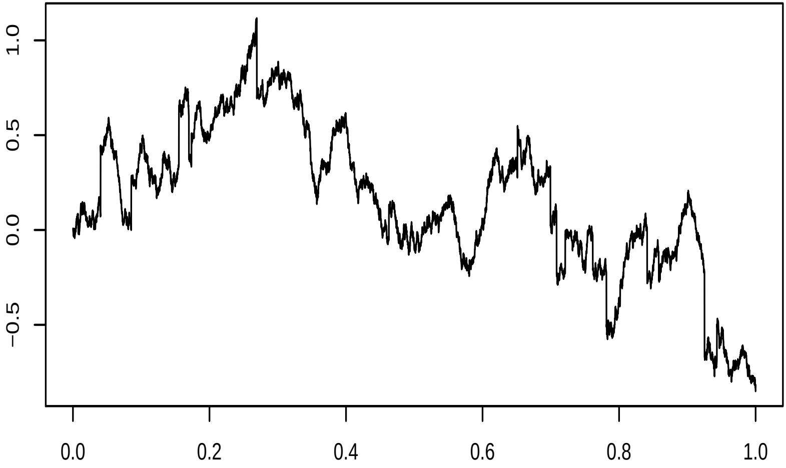
Preliminary considerations

- In the case of H_0: X is an MRW, the test statistics uses the unknown quantity ζ_4. In practice, it should been estimated from the data (cf. Abry *et al.*), which raises the question of the behaviour of the estimator of ζ_4 when X is Itō...
- We present some simulation results when the data is either a lognormal MRW, or a Brownian motion with some random jumps.
  In the case of a Brownian motion with no jumps, we can show that our test would need a huge number of data ($N >> 10^9$) to perform well, even when ζ_4 is known.

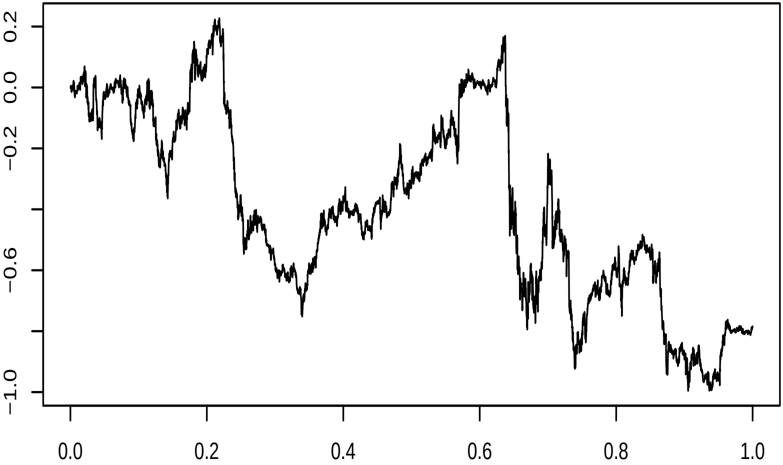
Choice of the Itō semimartingale

Standard Brownian motion with a few (≈ 30) large jumps uniformly distributed on $[-1/2, 1/2]$.



Choice of the MRW

Lognormal MRW with intermittency coefficient 0.025 ($\zeta_p = -0.025p^2 + 0.55p$).



Test results for 1 000 simulations of either an MRW, or an Itō semi-martingale

Rejection rates of $H_0 : X$ is an MRW when MRW's are simulated

	Level of the test		
N	90%	95%	99%
2048	9.6%	4.2%	1%
16384	8.9%	5.1%	1%
131072	9.4%	5%	0.7%

Rejection rates of $H_0 : X$ is an MRW when Itō semimartingales are simulated

	Level of the test		
N	90%	95%	99%
2048	43.6%	23.3%	4.4%
16384	92.4%	65%	16.6%
131072	99.6%	98.6%	40.5%

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