Empirical Mode Decomposition vs. wavelets for the analysis of scaling processes

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“Wavelets and Fractals”
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1. based on joint work with Gabriel Rilling, Paulo Gonçalvès & Patrice Abry
Wavelets and fractals: why and how

- Invariance under renormalized dilations
  - \textit{analyzed quantity} $\rightarrow$ fractal or scaling processes
  - \textit{analyzing tool} $\rightarrow$ wavelets and multiresolution

- Wedding wavelets and (multi-)fractals recognized long ago
  - Arneodo’s “mathematical microscope” + WTMM
  - F., Abry and Veitch’s “log-scale diagrams”
  - Jaffard’s “wavelet leaders”
  - multiscale models, etc.

Two main issues

1. assessing scaling
2. estimating exponents
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Discrete Wavelet Transform (DWT)

\[
\text{signal} = \text{approximation} + \text{detail} \\
&\text{iteration}
\]

- separation “approximation vs. detail” based on a priori (dyadic) filtering
- "global" analysis
- other schemes?
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Empirical Mode Decomposition (EMD) as an alternative

\[
\text{signal} = \text{fast oscillation} + \text{slow oscillation} \\
& \text{iteration}
\]

- separation “fast vs. slow” data driven
- "local" analysis based on extrema
- theoretical framework?

Empirical Mode Decomposition (EMD) as an alternative

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- *theoretical framework*

EMD algorithm

1. identify local maxima and local minima
2. deduce an upper envelope and a lower envelope by interpolation (cubic splines)
   - subtract the mean envelope from the signal
   - iterate until “mean envelope $= 0$" (*sifting*)
3. subtract the obtained mode from the signal
4. iterate on the residual

\[ x(t) = c_1(t) + r_1(t) \]
\[ = c_1(t) + c_2(t) + r_2(t) \]
\[ = \ldots \]
\[ = \sum_{k=1}^{K} c_k(t) + r_K(t), \]

with the $c_k(t)$'s referred to as *Intrinsic Mode Functions (IMFs)*
### EMD algorithm

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EMD algorithm in action

IMF 1; iteration 0
EMD algorithm in action

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EMD algorithm in action

IMF 1; iteration 0
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IMF 1; iteration 0

Graph showing the EMD algorithm in action.
EMD algorithm in action

IMF 1; iteration 0
EMD algorithm in action

IMF 1; iteration 0
EMD algorithm in action

IMF 1; iteration 0

residue
EMD algorithm in action

IMF 1; iteration 1
EMD algorithm in action

IMF 1; iteration 1
EMD algorithm in action

IMF 1; iteration 1
EMD algorithm in action

IMF 1; iteration 1
EMD algorithm in action
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IMF 1; iteration 1
EMD algorithm in action

IMF 1; iteration 1

residue
EMD algorithm in action

IMF 1; iteration 2
EMD algorithm in action

IMF 1; iteration 2
EMD algorithm in action

IMF 1; iteration 2
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IMF 1; iteration 2

residue
EMD algorithm in action

IMF 1; iteration 5

residue
EMD algorithm in action

IMF 2; iteration 2

Residue
EMD algorithm in action

IMF 3; iteration 14

residue
EMD algorithm in action

IMF 4; iteration 42

residue
EMD algorithm in action

IMF 5; iteration 13

residue
EMD algorithm in action

IMF 6; iteration 8

residue
EMD algorithm in action

IMF 7; iteration 21

residue
EMD algorithm in action

Empirical Mode Decomposition

signal

imf1

imf2

imf3

imf4

imf5

imf6

imf7

imf8

res.
EMD algorithm in action
EMD algorithm in action

signal

mode #1

time

frequency

signal

mode #1

time

frequency
EMD algorithm in action
EMD algorithm in action

signal

mode #1

mode #2

mode #3
EMD features

- *Locality* — The method operates at the scale of one oscillation
- *Adaptivity* — The decomposition is fully data-driven
- *Multiresolution* — The iterative process explores sequentially the “natural” constitutive scales of a signal
- *Oscillations of any type* — No assumption on the (e.g., harmonic) nature of oscillations ⇒ 1 nonlinear oscillation = 1 mode
- *No analytic definition* — The decomposition is only defined as the output of an algorithm ⇒ analysis and evaluation?
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EMD vs. wavelets

Similarity: both achieve a decomposition into “fluctuations” and “trend”

\[ x(t) = \sum_k c_k(t) + r_k(t) \quad \text{(EMD)} \]

\[ = \sum_k d_k(t) + a_k(t) \quad \text{(DWT)} \]

with \( d_k(t) = \sum_n \langle x, \psi_{kn} \rangle \psi_{kn}(t) \)

and \( a_k(t) = \sum_n \langle x, \varphi_{kn} \rangle \varphi_{kn}(t) \)

Difference: scales are pre-determined for DWT

\((\{\varphi, \psi\}_{kn}(t) = 2^{-k/2} \{\varphi, \psi\}(2^{-k} t - n))\) and adaptive (data-driven) for EMD
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EMD and fractals

- the example of fractional Gaussian noise (fGn)
- “spontaneous”, almost dyadic, filterbank organization

\[ S_{k',H}(f) \approx 2^{(2H-1)(k'-k)} S_{k,H}(2^{k'}-k f) \]

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\[
S_{k',H}(f) \approx 2^{(2H-1)(k'-k)} S_{k,H} \left(2^{k'} - kf \right)
\]

Interpretation

1. **time-domain**: EMD “impulse response”

2. **frequency domain**: 2-tone example
One or two components?

\[
\cos(\omega_1 t) + \cos(\omega_2 t) = 2 \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)
\]
One or two components?

\[ \cos(\omega_1 t) + \cos(\omega_2 t) = 2 \cos \left( \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \]
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\]
Simulations

Signal

\[ x(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t + \varphi), \quad f_1 > f_2 \]

Analysis of its EMD

- only the first IMF is computed: if separation, it should be equal to the highest frequency component \( x_1(t) \)
- criterion (\( = 0 \) if separation): 
  \[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

- sampling effects are neglected: \( f_1, f_2 \ll f_s \), with \( f_s \) the sampling frequency
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\[ x_1(t) \quad \text{and} \quad x_2(t) \]

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Signal

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\[ x_1(t) \]

\[ x_2(t) \]

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\]

- sampling effects are neglected: \( f_1, f_2 \ll f_s \), with \( f_s \) the sampling frequency
Sum of two tones

\( \frac{f_2}{f_1} = 0.08, \frac{a_2}{a_1} = 0.33 \)

\[
\begin{align*}
\left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) &= \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \\
&= 0 \quad \text{if separation}
\end{align*}
\]
Sum of two tones

$$\frac{f_2}{f_1} = 0.13, \frac{a_2}{a_1} = 0.33$$

$$c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2}$$

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.17, \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.22, \quad \frac{a_2}{a_1} = 0.33 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} = 0 \quad \text{if separation}
\]
sum of two tones

\[ \frac{f_2}{f_1} = 0.26, \quad \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.30, \quad \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.35, \frac{a_2}{a_1} = 0.33 \]

\[ x_2 \text{signal IMF 1 reste} \]

\[ \log_{10} \frac{a_2}{a_1} \]

\[ \frac{f_2}{f_1} \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.39, \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\left\| IMF_1(t) - x_1(t) \right\|_2}{\left\| x_2(t) \right\|_2} \]

= 0 if separation
Sum of two tones

\[
f_2/f_1 = 0.44, \quad a_2/a_1 = 0.33
\]

\[
x_1
\]

\[
x_2
\]

\[
\text{IMF 1 signal}
\]

\[
\text{reste}
\]

\[
\log_{10} a_2/a_1
\]

\[
f_2/f_1
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}
\]

\[
= 0 \quad \text{if separation}
\]
Sum of two tones

\[
f_2/f_1 = 0.48, \quad a_2/a_1 = 0.33
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}}
\]

= 0 \quad \text{if separation}
Sum of two tones

\[ f_2/f_1 = 0.52, \quad a_2/a_1 = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{L^2}}{\| x_2(t) \|_{L^2}} \]

= 0 \quad \text{if separation}
Sum of two tones

\[ \frac{f_2}{f_1} = 0.57, \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell^2}}{\| x_2(t) \|_{\ell^2}} \]

= 0 if separation
Sum of two tones

\[ f_2/f_1 = 0.61, \quad a_2/a_1 = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}} \]

= 0 if separation
Sum of two tones

\[
\frac{f_2}{f_1} = 0.66, \quad \frac{a_2}{a_1} = 0.33
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}
\]

= 0 \quad \text{if separation}
Sum of two tones

\[ \frac{f_2}{f_1} = 0.70, \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[
f_2/f_1 = 0.74, \quad a_2/a_1 = 0.33
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}
\]

\[
= 0 \quad \text{if separation}
\]
**Sum of two tones**

\[ \frac{f_2}{f_1} = 0.79, \frac{a_2}{a_1} = 0.33 \]

\[
\begin{array}{c}
\text{x}_1 \\
\text{x}_2 \\
\text{IMF 1 signal} \\
\text{reste}
\end{array}
\]

\[
c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}
\]

\[= 0 \quad \text{if separation} \]
Sum of two tones

\[
f_2/f_1 = 0.83, \ a_2/a_1 = 0.33
\]

\[
\begin{align*}
&c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \\
= 0 & \text{ if separation}
\end{align*}
\]
Sum of two tones

$f_2/f_1 = 0.88, \ a_2/a_1 = 0.33$

$\begin{align*}
\text{x}_1 \\
\text{x}_2 \\
\text{IMF 1 signal} \\
\text{reste}
\end{align*}$

$$c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2}$$

$= 0$ if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.92, \quad \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.08, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ f_2/f_1 = 0.13, \quad a_2/a_1 = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

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Sum of two tones

\[ \frac{f_2}{f_1} = 0.17, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.22, \quad \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.26, \quad \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell^2}}{\| x_2(t) \|_{\ell^2}} \]

= 0 \text{ if separation}
Sum of two tones

\[
f_2/f_1 = 0.30, \quad a_2/a_1 = 3.00
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}
\]

= 0 \quad \text{if separation}
Sum of two tones

\[ \frac{f_2}{f_1} = 0.35, \frac{a_2}{a_1} = 3.00 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}
\]

\[ = 0 \text{ if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.39, \quad \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2} \]

= 0 if separation
Sum of two tones

\[
f_2/f_1 = 0.44, \ a_2/a_1 = 3.00
\]

\[
c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2}
\]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.48, \ \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.52, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{L^2}}{\| x_2(t) \|_{L^2}} \]

= 0 if separation
**Sum of two tones**

\[
\frac{f_2}{f_1} = 0.57, \quad \frac{a_2}{a_1} = 3.00
\]

\[c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}}
\]

\[= 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.61, \quad \frac{a_2}{a_1} = 3.00 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.66, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} = 0 \text{ if separation} \]
**Sum of two tones**

\[ \frac{f_2}{f_1} = 0.70, \quad \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.74, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.79, \frac{a_2}{a_1} = 3.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

= 0 if separation
### Sum of two tones

\[ \frac{f_2}{f_1} = 0.83, \frac{a_2}{a_1} = 3.00 \]

\[
\begin{align*}
x_1 & \quad \text{signal} \\
x_2 & \\
imf_1 & \text{signal} \\
\text{reste} & \quad \text{signal}
\end{align*}
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} = 0 \quad \text{if separation}
\]
**Sum of two tones**

\[
f_2/f_1 = 0.88, \quad a_2/a_1 = 3.00
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.92, \quad \frac{a_2}{a_1} = 3.00 \]

\[
\begin{align*}
    x_1 & \quad \text{\textcolor{blue}{signal}} \\
    x_2 & \quad \text{\textcolor{blue}{IMF 1 signal}} \\
    \text{reste} & \quad \text{\textcolor{blue}{IMF 1}}
\end{align*}
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ f_2/f_1 = 0.44, \quad a_2/a_1 = 0.20 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

= 0 \quad \text{if separation}
Sum of two tones

\[ f_2/f_1 = 0.44, \quad a_2/a_1 = 0.24 \]

c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}

= 0 \quad \text{if separation}
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.28 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.33 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.39 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.47 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.55 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} \]

= 0 \quad \text{if separation}
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.65 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.78 \]

\[
\begin{align*}
\text{Signal} & \\
\text{Rest} & \\
\text{IMF 1} & \\
\end{align*}
\]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|\text{IMF}_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 0.92 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 1.09 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\left\| IMF_1(t) - x_1(t) \right\|_{l_2}}{\left\| x_2(t) \right\|_{l_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 1.29 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}
\]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 1.53 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 1.81 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_2}{\| x_2(t) \|_2} = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 2.14 \]

\[
c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\|IMF_1(t) - x_1(t)\|_2}{\|x_2(t)\|_2} = 0 \quad \text{if separation}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 2.54 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} 
= 0 \quad \text{if separation} \]
Sum of two tones

\[ f_2/f_1 = 0.44, \quad a_2/a_1 = 3.01 \]

\[
\begin{align*}
\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi & = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \\
& = 0 \quad \text{if separation}
\end{align*}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 3.56 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

= 0 if separation
Sum of two tones

\[ f_2/f_1 = 0.44, \quad a_2/a_1 = 4.22 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

\[ = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 5.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} = 0 \quad \text{if separation} \]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 5.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \]

= 0 if separation
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 5.00 \]

\[
\begin{align*}
\text{c} \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = & \quad \frac{\| \text{IMF}_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}} \\
& = 0 \quad \text{if separation}
\end{align*}
\]
Sum of two tones

\[ \frac{f_2}{f_1} = 0.44, \quad \frac{a_2}{a_1} = 5.00 \]

\[ c \left( \frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi \right) = \frac{\| \text{IMF}_1(t) - x_1(t) \|_{l_2}}{\| x_2(t) \|_{l_2}} \]

\[ = 0 \quad \text{if separation} \]
Scaling estimation

- Multiresolution decomposition of $x(t)$ into scale-dependent contributions $x_k(t)$
- Evaluation of fluctuation properties (e.g., variance) as a function of scale $k$
- Estimation of Hurst exponent from slope $\alpha$ in a log-log plot

$fBm/fGn : \alpha = 2H \pm 1$
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\[ \text{fBm/fGn : } \alpha = 2H \pm 1 \]
EMD vs. wavelets for scaling estimation

Wavelet analysis based on log-scale diagrams

\[ \log_2 V[k] = \alpha k + C, \]

with \( V[k] = \text{var} \, d_k[n] \), estimated by \( \hat{V}[k] = d_k^2[n] \), as a function of the fixed scale \( k \)

EMD analysis based on “modegrams” \( (d_k[n] \rightarrow c_k[n]) \), with variance a function of either the IMF index \( k \) or the natural scale given by the mean period \( T[k] \), estimated as

\[ \hat{T}[k] := \frac{\text{distance between the first and the last zero-crossing}}{\text{number of zero-crossings} - 1} \]

Remark: implicit trend removal, as for wavelets with enough vanishing moments
EMD vs. wavelets for scaling estimation

1. Wavelet analysis based on log-scale diagrams

\[ \log_2 V[k] = \alpha k + C, \]

with \( V[k] = \text{var } d_k[n], \) estimated by \( \hat{V}[k] = d_k^2[n], \) as a function of the fixed scale \( k \)

2. EMD analysis based on “modegrams” (\( d_k[n] \rightarrow c_k[n] \)), with variance a function of either the IMF index \( k \) or the natural scale given by the mean period \( T[k], \) estimated as

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EMD vs. wavelets for scaling estimation

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*Remark*: implicit trend removal, as for wavelets with enough vanishing moments.
Modegrams of fGn
Dimension estimation

- Multiresolution decomposition of $x(t)$ into scale-dependent approximations $a_k(t)$
- Evaluation of the length of the corresponding graph
  \[
  \mathcal{L}(a_k) = \int \sqrt{1 + \dot{a}_k^2(t)} \, dt
  \]
- Estimation of (regularization) dimension $D_x$ from slope in a log-log plot (small-scale limit)

\[
\text{fBm : } D_x = 2 - H
\]

[Lévy-Vehel & Roueff, 1998]
Dimension estimation

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fBm: D_x = 2 - H
\]

[Lévy-Vehel & Roueff, 1998]
EMD vs. wavelets for regularization dimension estimation

**DWT**

\[ \mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_n \langle x, \varphi_{kn} \rangle \varphi_{kn}(t) \right|^2} \, dt \]

**EMD**

\[ \mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_{k' \geq k} c_{k'}(t) \right|^2} \, dt \]
EMD vs. wavelets for regularization dimension estimation

DWT

\[
\mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_n \langle x, \varphi_{kn} \rangle \varphi_{kn}(t) \right|^2} \, dt
\]

EMD

\[
\mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_{k' \geq k} c_{k'}(t) \right|^2} \, dt
\]
EMD vs. wavelets for regularization dimension estimation

1. DWT

\[ \mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_n \langle x, \varphi_{kn} \rangle \varphi_{kn}(t) \right|^2} \; dt \]

2. EMD

\[ \mathcal{L}(a_k) = \int \sqrt{1 + \left| \frac{d}{dt} \sum_{k' \geq k} c_{k'}(t) \right|^2} \; dt \]
Experimental comparison

- **Synthetic fBm**
  - circulant matrix method of [Wood & Chan, 1994]
  - $H = 0.1, 0.2, \ldots, 0.9$
  - 10,000 realizations of 1024 data points

- comparison of 6 estimators
  - 3 variance-based: DWT, wavelet leaders and EMD
  - 3 length-based: smoothed graph, DWT and EMD

- **Weighted** linear regressions in log-log plots

- Performance measured by mean-square error and distribution of estimates

[Gonçalvès et al., 2007]
Mean Square Error

\[ \text{MSE} = (\mathbb{E} \hat{H} - H)^2 + \text{var} \hat{H} \]

for \textit{variance-based} (DWT, wavelet leaders (WL) and EMD) and \textit{length-based} (smoothed graph (G), DWT and EMD) estimators.

<table>
<thead>
<tr>
<th></th>
<th>DWT</th>
<th>WL</th>
<th>EMD</th>
<th>G</th>
<th>DWT</th>
<th>EMD</th>
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<tr>
<td>(H = 0.1)</td>
<td>0.0439</td>
<td>0.0044</td>
<td>0.0194</td>
<td>0.0497</td>
<td>0.0057</td>
<td>0.0135</td>
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<td>(H = 0.2)</td>
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<td>0.0099</td>
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<td>0.0058</td>
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<tr>
<td>(H = 0.3)</td>
<td>0.0090</td>
<td>0.0051</td>
<td>0.0074</td>
<td>0.0062</td>
<td>0.0079</td>
<td>0.0023</td>
</tr>
<tr>
<td>(H = 0.4)</td>
<td>0.0060</td>
<td>0.0059</td>
<td>0.0083</td>
<td>0.0030</td>
<td>0.0102</td>
<td>0.0016</td>
</tr>
<tr>
<td>(H = 0.5)</td>
<td>0.0048</td>
<td>0.0049</td>
<td>0.0069</td>
<td>0.0019</td>
<td>0.0123</td>
<td>0.0016</td>
</tr>
<tr>
<td>(H = 0.6)</td>
<td>0.0044</td>
<td>0.0041</td>
<td>0.0076</td>
<td>0.0015</td>
<td>0.0139</td>
<td>0.0017</td>
</tr>
<tr>
<td>(H = 0.7)</td>
<td>0.0042</td>
<td>0.0034</td>
<td>0.0088</td>
<td>0.0016</td>
<td>0.0163</td>
<td>0.0018</td>
</tr>
<tr>
<td>(H = 0.8)</td>
<td>0.0040</td>
<td>0.0031</td>
<td>0.0117</td>
<td>0.0020</td>
<td>0.0181</td>
<td>0.0019</td>
</tr>
<tr>
<td>(H = 0.9)</td>
<td>0.0041</td>
<td>0.0030</td>
<td>0.0888</td>
<td>0.0027</td>
<td>0.0175</td>
<td>0.0012</td>
</tr>
</tbody>
</table>
Distribution of estimates
Facts on EMD-based estimations

1. **compare favorably** with more “classical” estimations
2. **length** better than variance
3. better than wavelets on **short data records**
4. adapt automatically to “natural” scales ⇒ easier identification of “inertial” range and **assessment of scaling**
5. **robustness** to superimposed periodic components and possible **extensions**

[Huang, Schmitt et al., 2008-2009]
Concluding remarks

- **“Classical” EMD**
  - data-driven multiresolution decomposition ⇒ adaptation to natural scales (scaling assessment and estimation)
  - empirical approach ⇒ no firm theoretical framework

- Two possible ways out
  - work towards a comprehensive theory of current EMD (e.g., PDE formulations)
  - consider more tractable variations keeping the flavour of EMD (e.g., “synchrosqueezing” of Daubechies et al.)
Fig. 3: Scaling exponents $\xi(q)$ for fractional Brownian motion simulations with $H = 0.2$, 0.4, 0.6 and 0.8, respectively.

[Huang, Schmitt et al., 2008]
EMD and multifractals

Figure 7. Comparaison de l’estimation de l’exposant multifractal à l’aide des fonctions de structure et à l’aide de l’analyse spectrale de Hilbert d’ordre quelconque pour des simulations multifractales lognormales avec $\mu = 0.05$.

[Huang, Schmitt et al., 2009]
Turbulence data

Fig. 2: (a) Mean frequency vs. mode number for the turbulent velocity time series. There is an exponential decrease with a slope very close to 1. This indicates that EMD acts as a filter bank which is almost dyadic. (b) Fourier spectrum of each mode (from 1 to 12) showing that they are narrow-banded. The slope of the reference line is $-5/3$ corresponding to the inertial-range Kolmogorov spectrum.

[Huang, Schmitt et al., 2008]
Turbulence data

Fig. 8: Comparison of the scaling exponents $\xi(q) - 1$ (diamond) with the classical $\zeta(q)$ obtained from structure functions analysis with the ESS method (dashed line) and K41 $q/3$ (solid line). The insert shows the departure from the K41 law.

[Huang, Schmitt et al., 2008]