Empirical Mode Decomposition vs. wavelets for the analysis of scaling processes

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"Wavelets and Fractals"
Esneux (B) — April 26-28, 2010

1. based on joint work with Gabriel Rilling, Paulo Gonçalvès & Patrice Abry

Wavelets and fractals : why and how

- Invariance under renormalized dilations
  - *analyzed quantity* $\rightarrow$ fractal or scaling processes
  - *analyzing tool* $\rightarrow$ wavelets and multiresolution
- Wedding wavelets and (multi-)fractals recognized long ago
  - Arneodo's "mathematical microscope" + WTMM
  - F., Abry and Veitch's "log-scale diagrams"
  - Jaffard's "wavelet leaders"
  - multiscale models, etc.

Two main issues

1. assessing scaling
2. estimating exponents

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Discrete Wavelet Transform (DWT)

$$\boxed{\begin{array}{c}\text{signal} = \text{approximation} + \text{detail} \\ \& \\ \text{iteration}\end{array}}$$

- separation "approximation vs. detail" based on a priori (dyadic) filtering
- "global" analysis
- *other schemes ?*

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Empirical Mode Decomposition (EMD) as an alternative

$$\boxed{\begin{array}{c}\text{signal = fast oscillation + slow oscillation}\\ \&\\ \text{iteration}\end{array}}$$

- separation “fast vs. slow” data driven
- "local" analysis based on extrema
- *theoretical framework ?*

[Huang *et al.*, *Proc. Roy. Soc. A.*, 1998]

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EMD algorithm

1. identify local maxima and local minima
2. deduce an upper envelope and a lower envelope by interpolation (cubic splines)
   1. subtract the mean envelope from the signal
   2. iterate until “mean envelope $= 0$" (*sifting*)
3. subtract the obtained mode from the signal
4. iterate on the residual

$$\begin{aligned} x(t) &= c_1(t) + r_1(t) \\ &= c_1(t) + c_2(t) + r_2(t) \\ &= \ldots \\ &= \sum_{k=1}^{K} c_k(t) + r_K(t), \end{aligned}$$

with the $c_k(t)$'s referred to as *Intrinsic Mode Functions* (IMFs)

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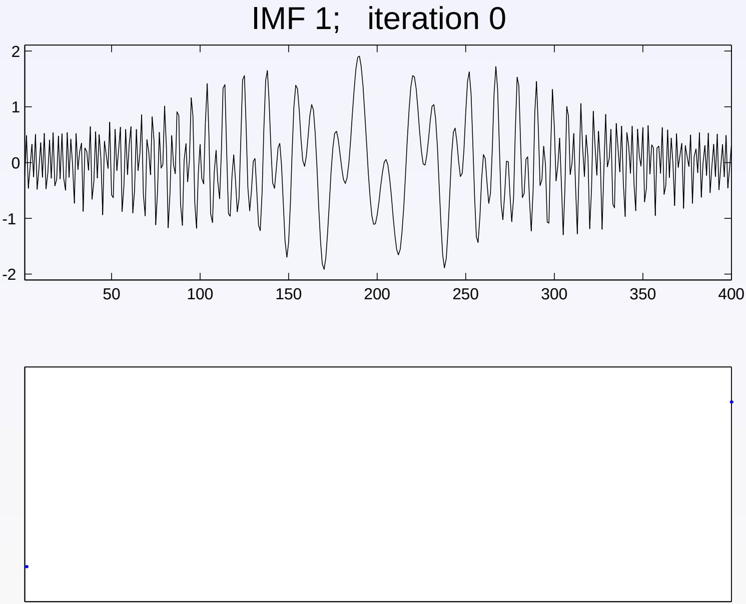

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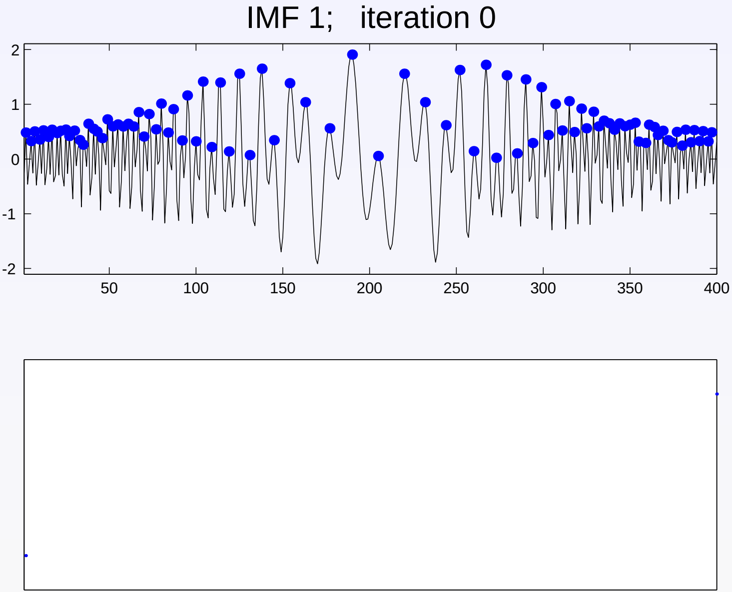

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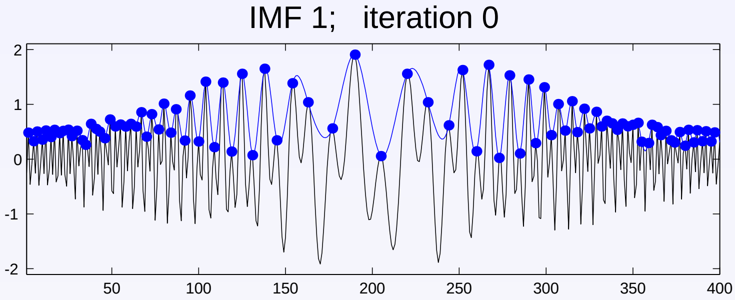

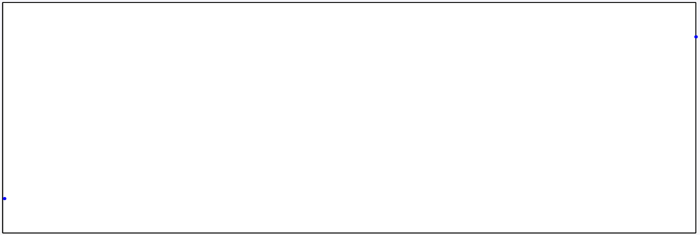

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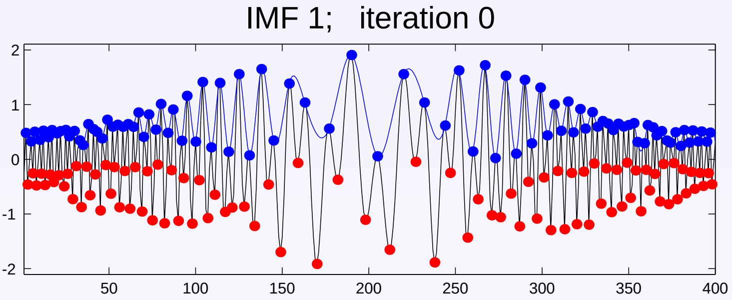

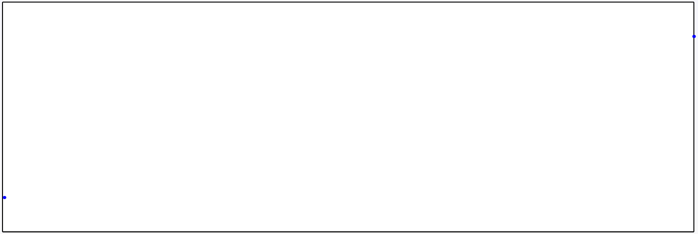

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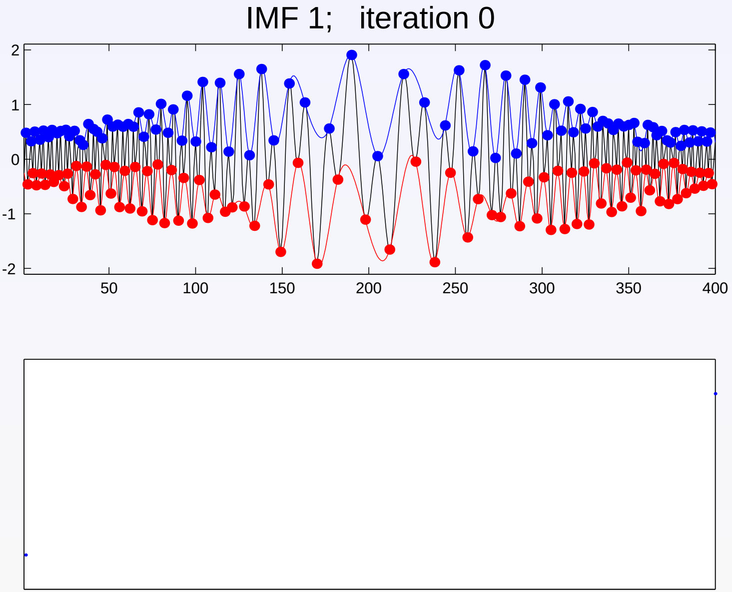

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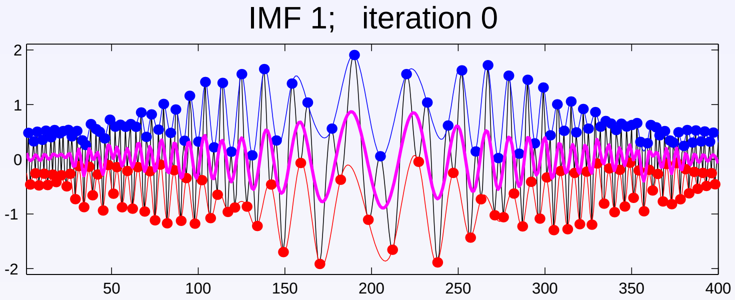

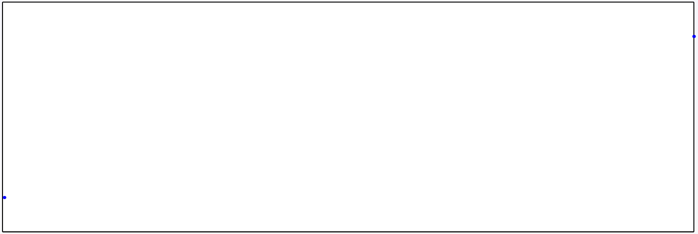

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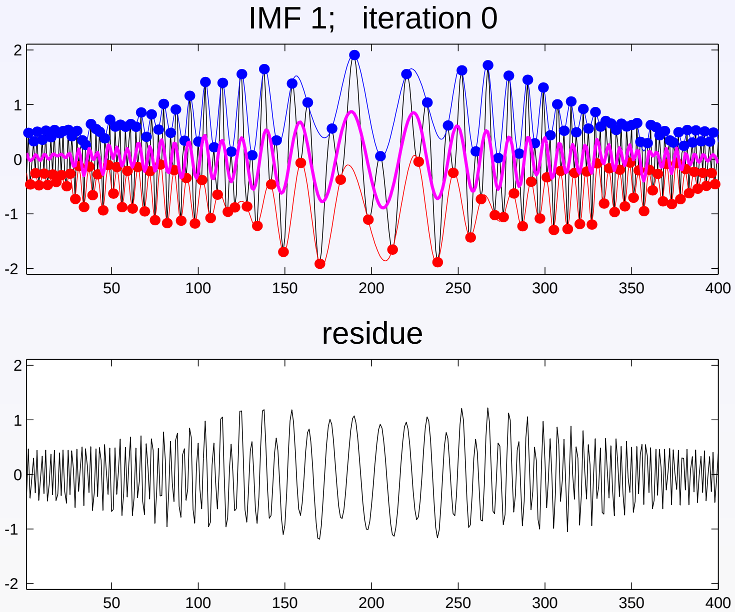

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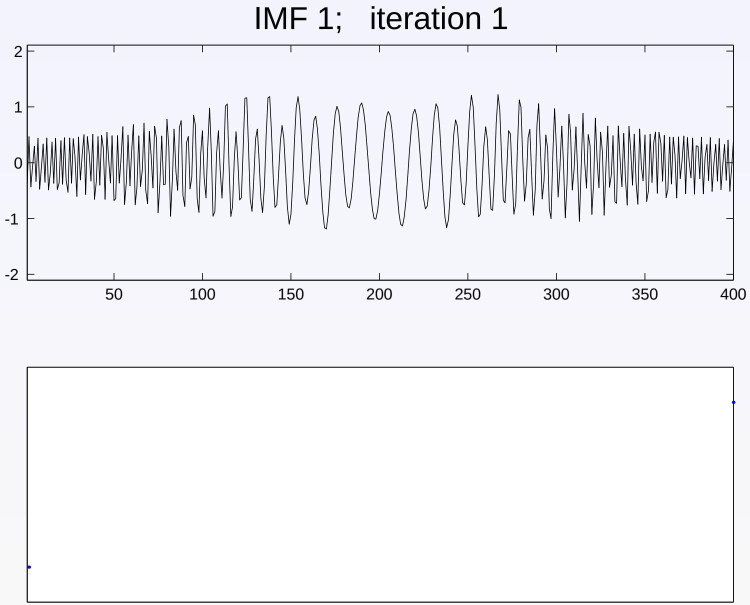

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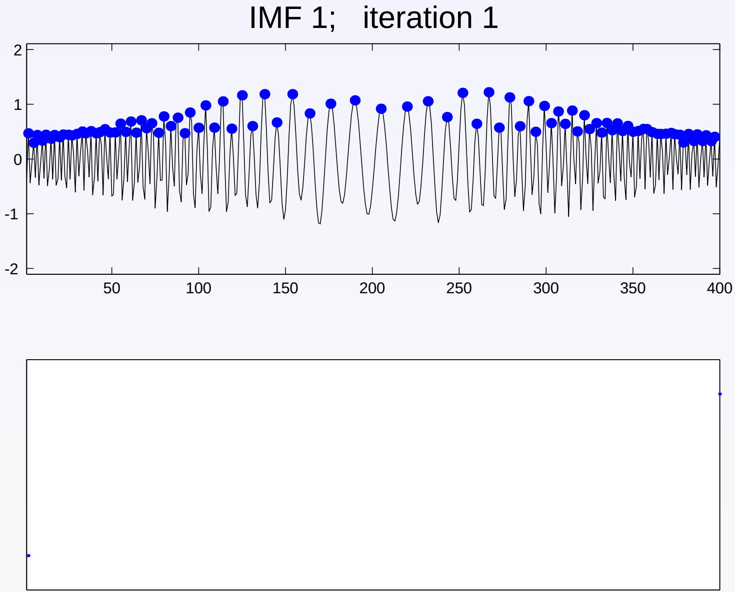

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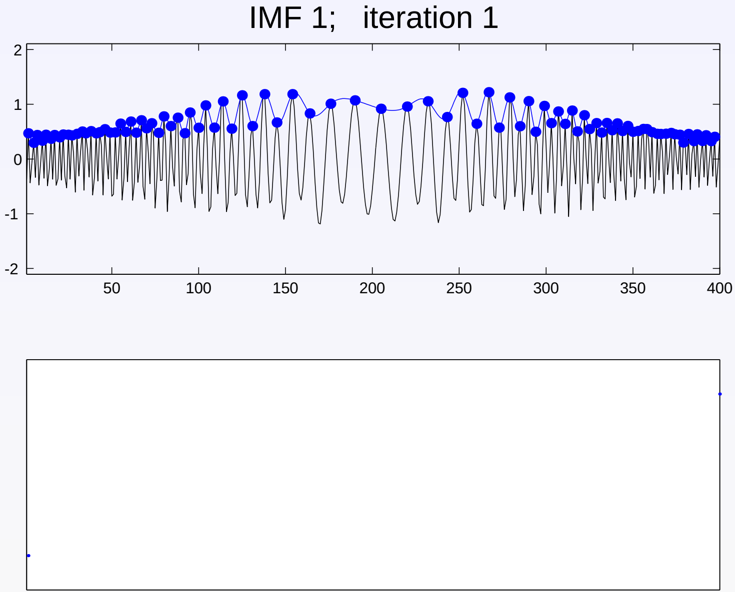

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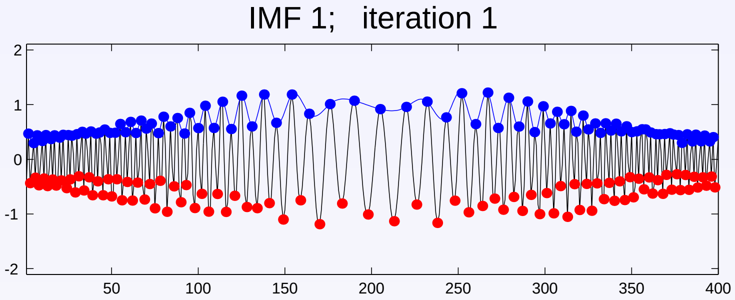

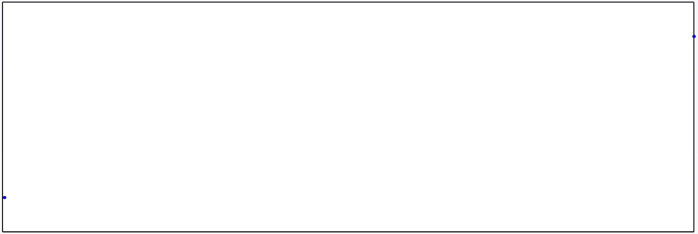

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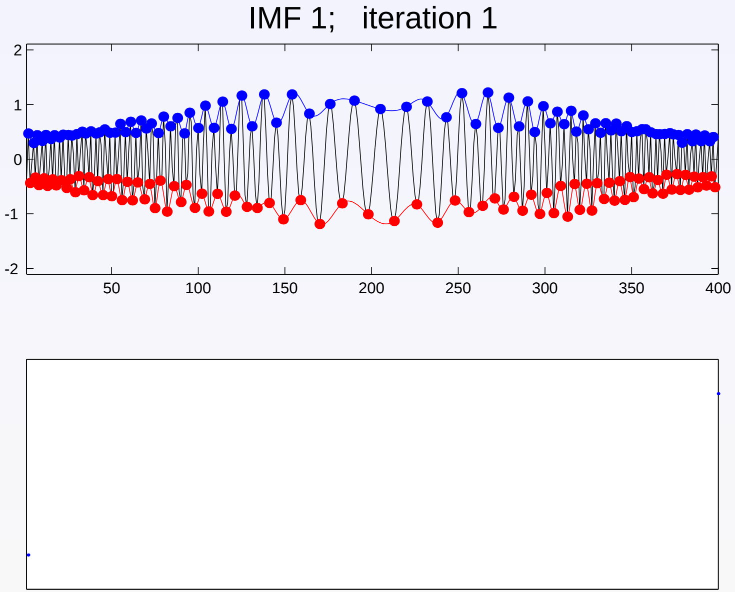

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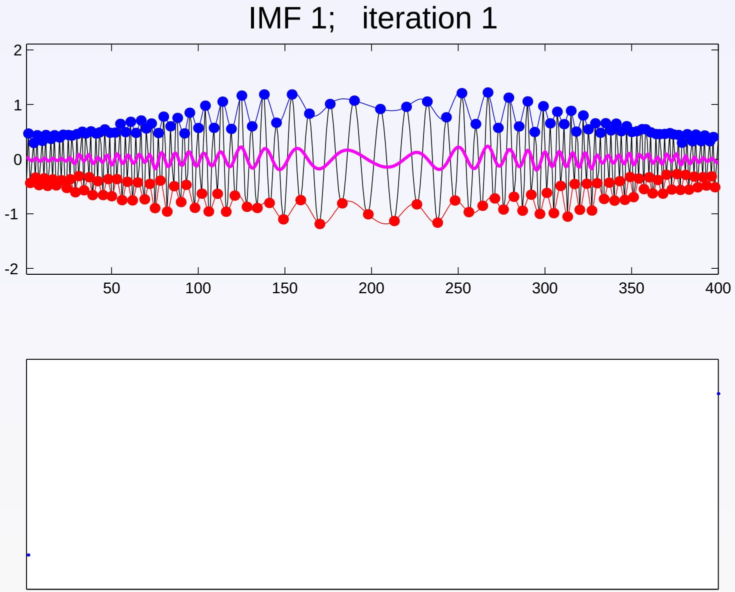

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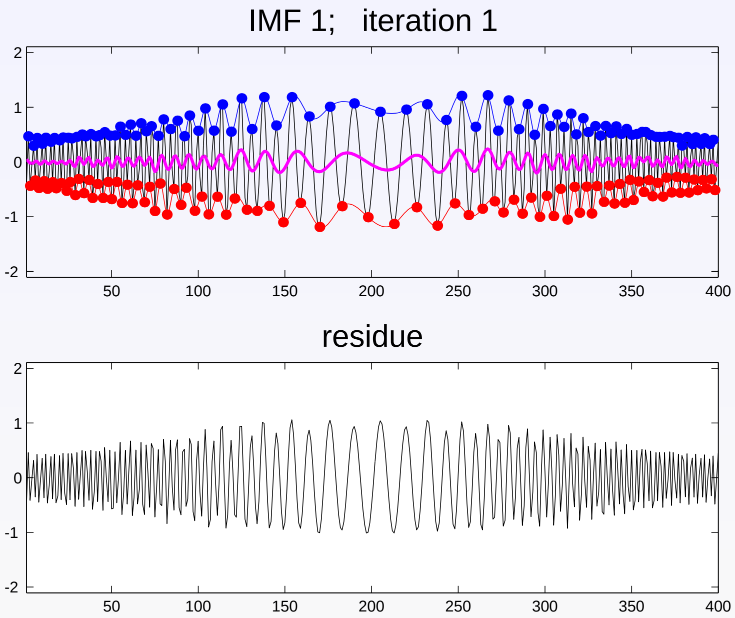

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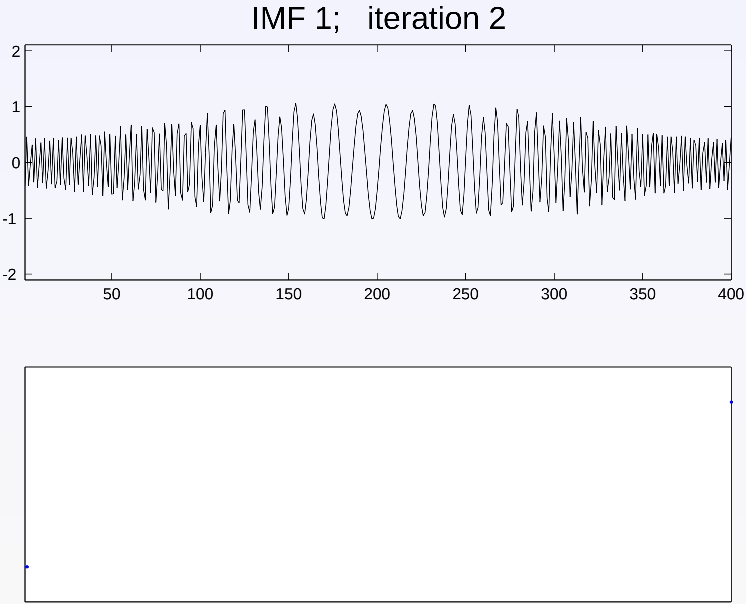

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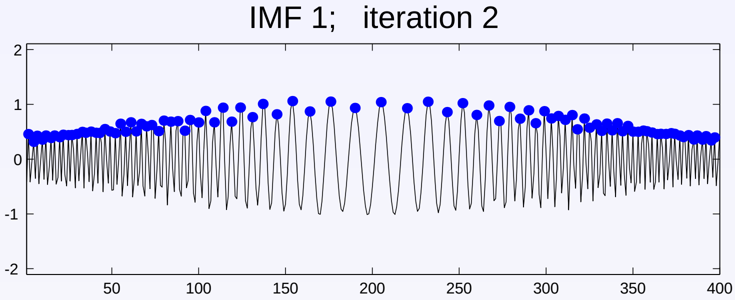

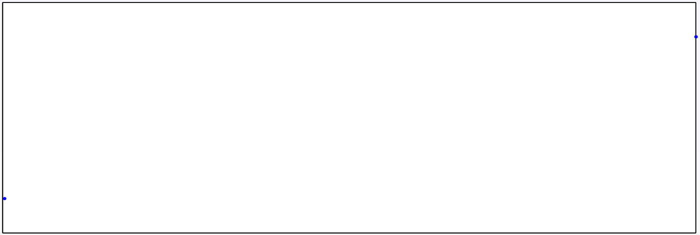

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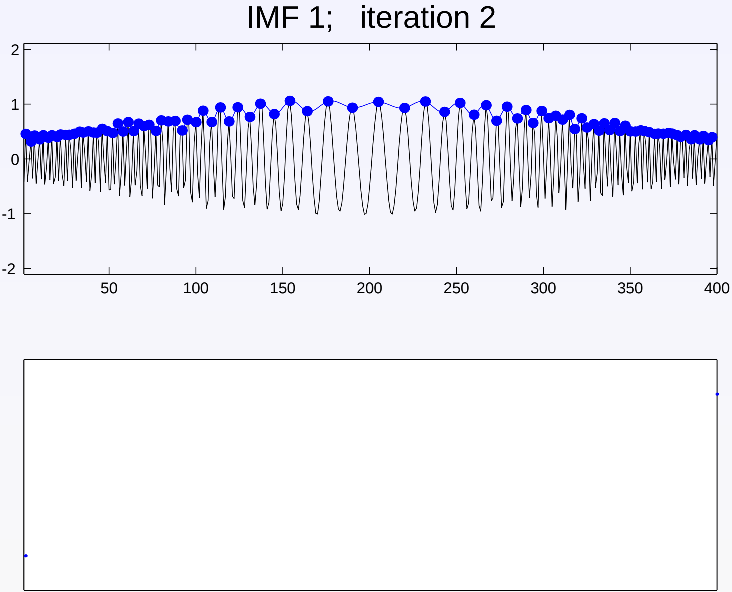

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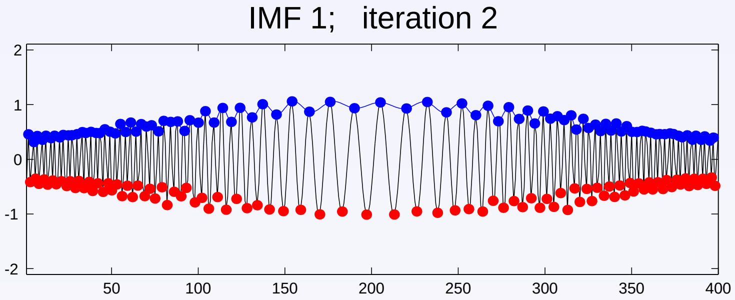

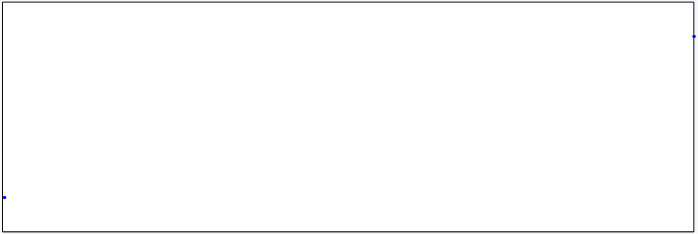

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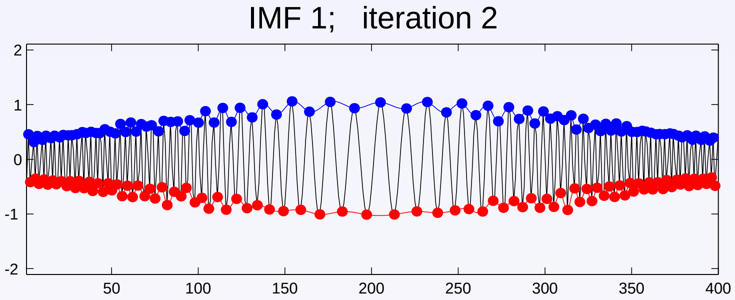

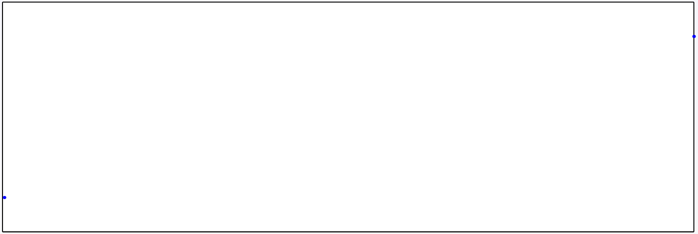

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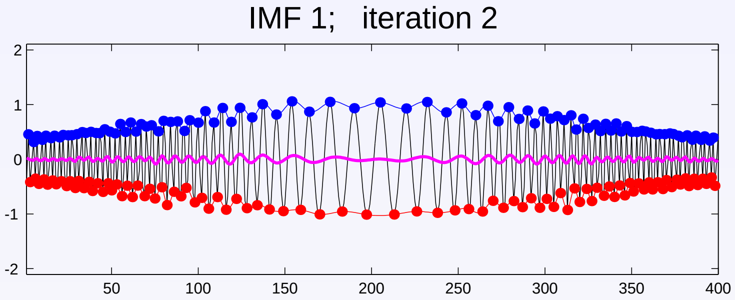

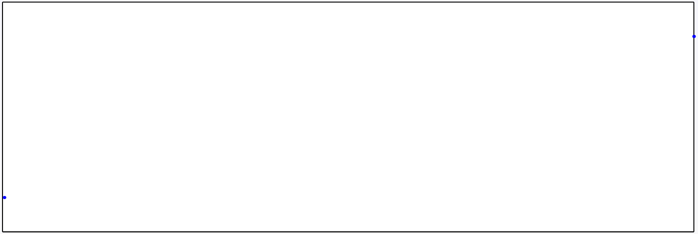

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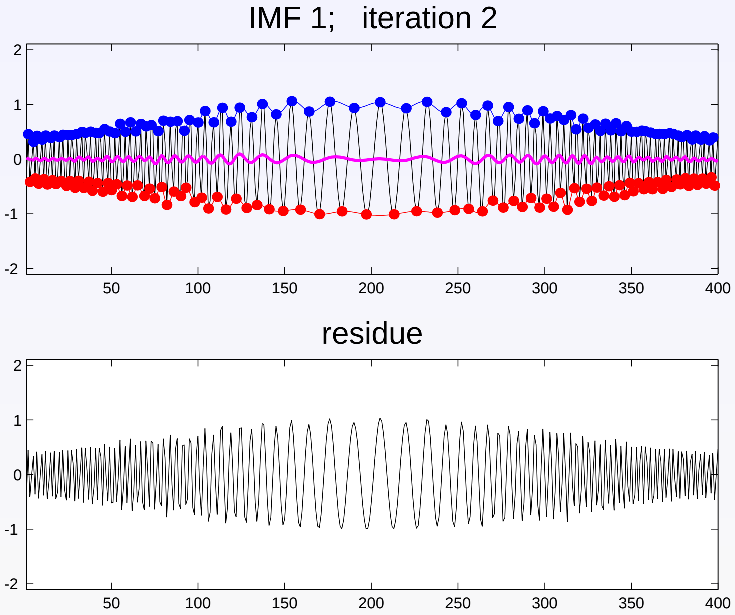

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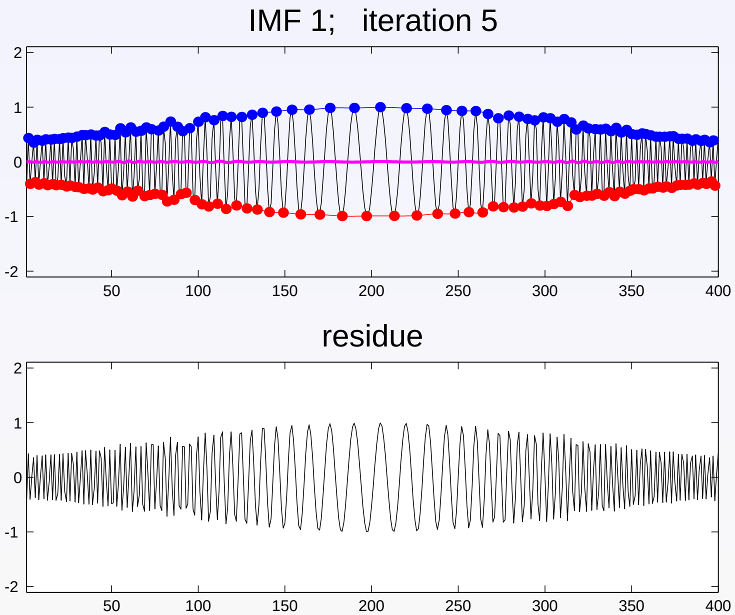

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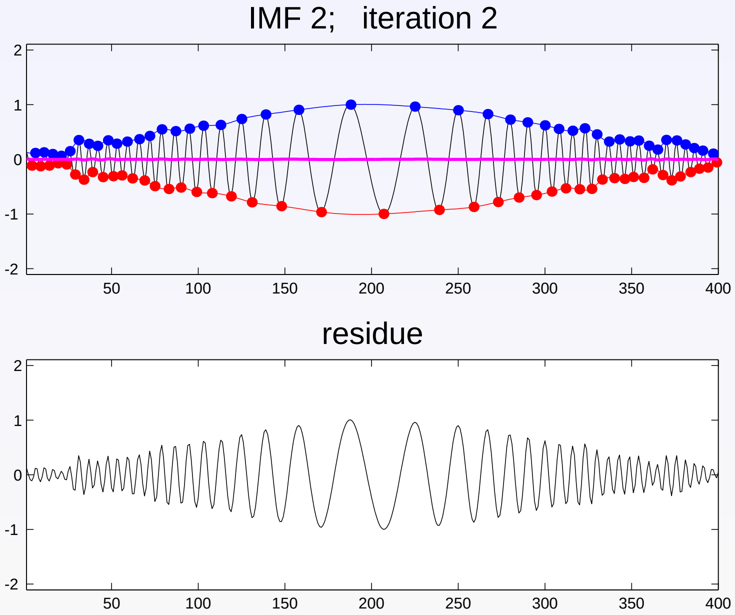

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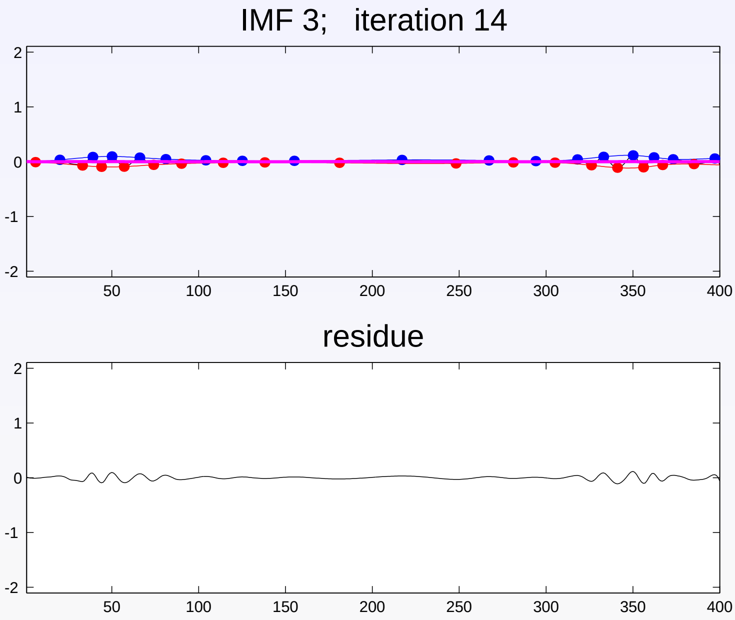

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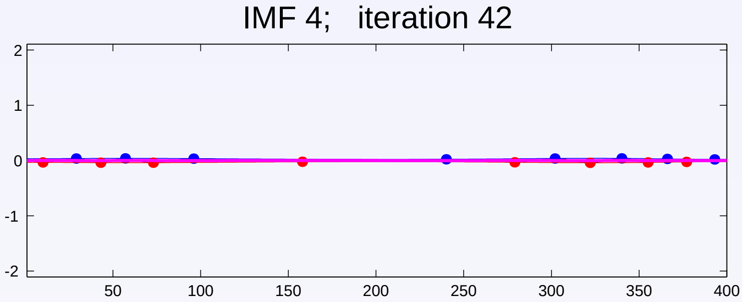

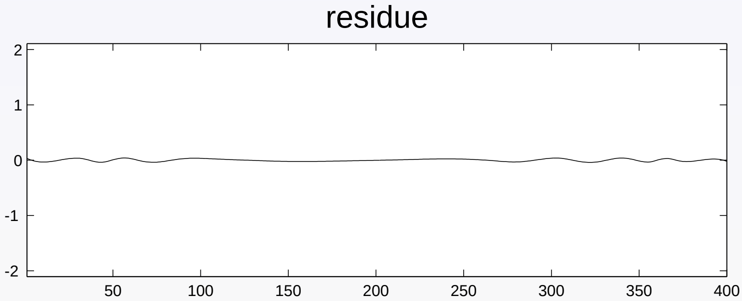

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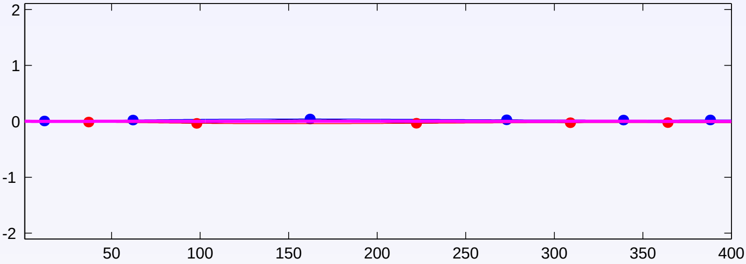
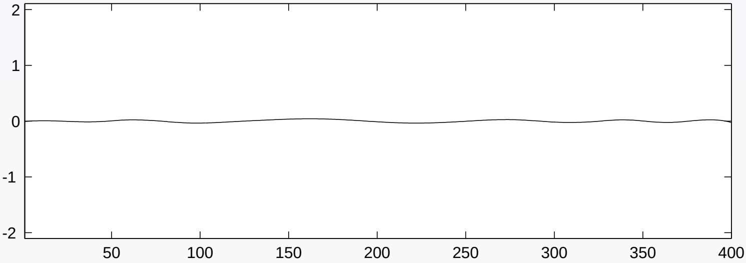
IMF 5;   iteration 13



residue



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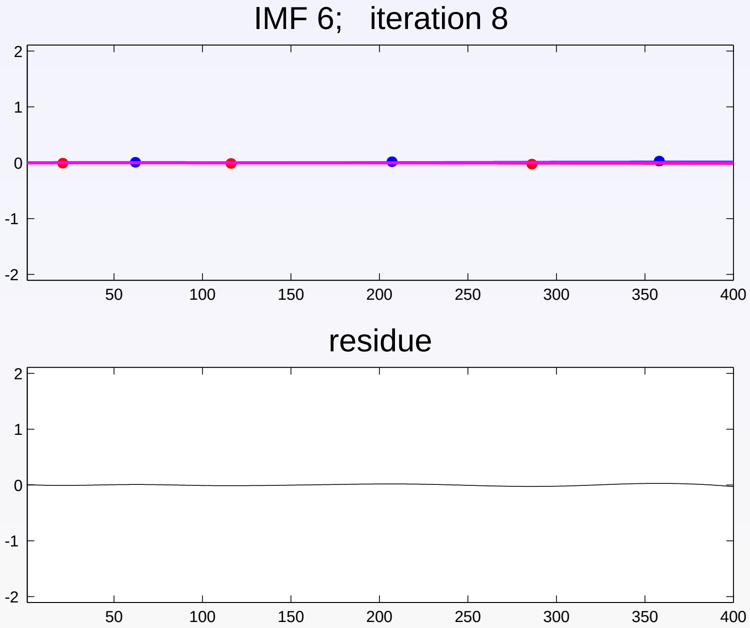

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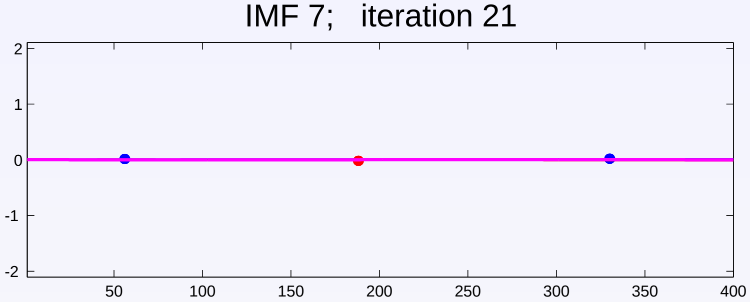

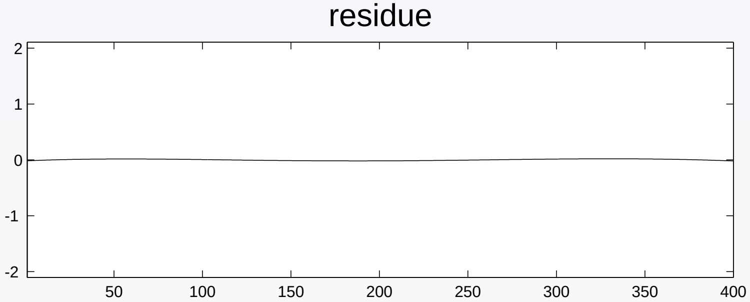

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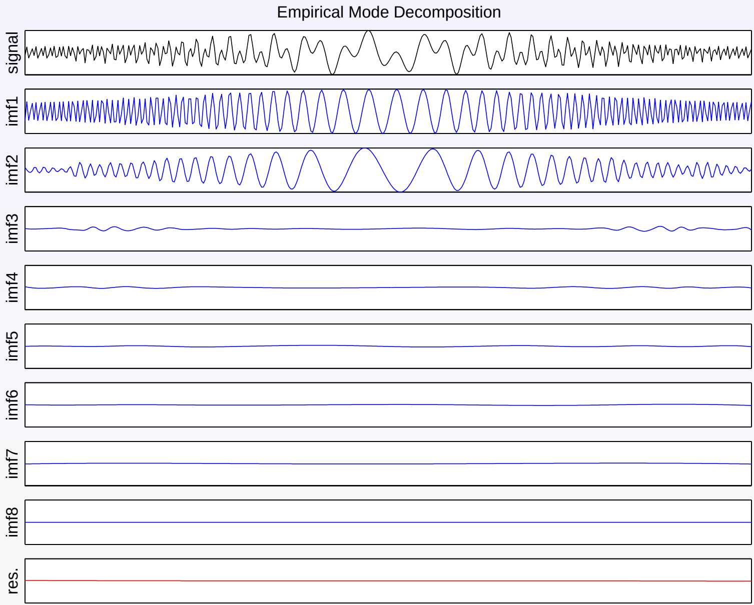

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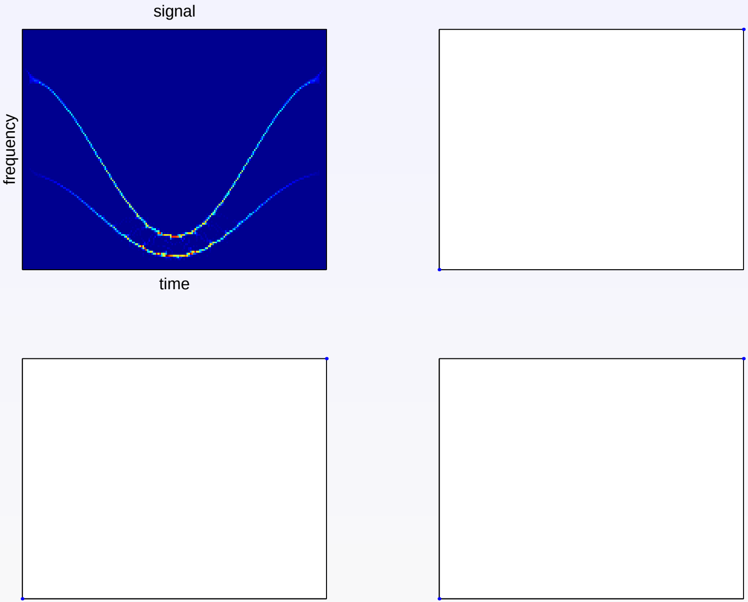

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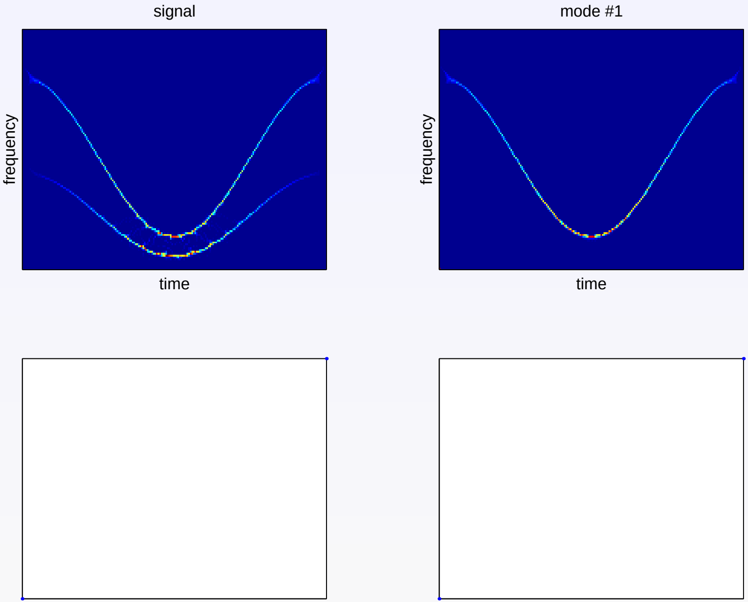

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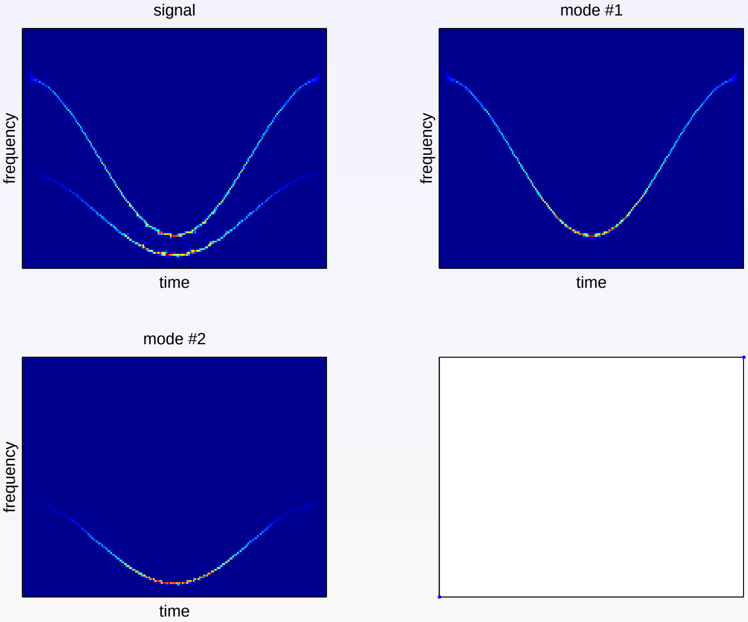

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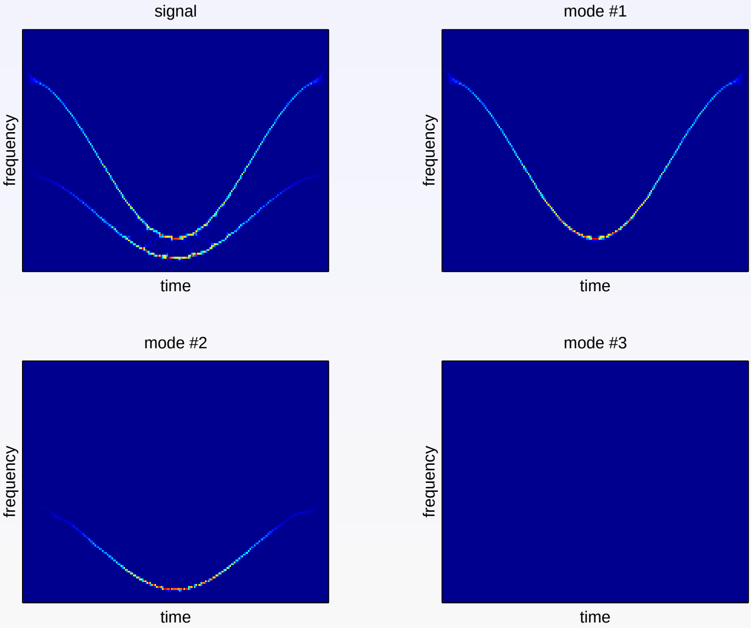

EMD features

- *Locality* — The method operates at the scale of one oscillation
- *Adaptivity* — The decomposition is fully data-driven
- *Multiresolution* — The iterative process explores sequentially the "natural" constitutive scales of a signal
- *Oscillations of any type* — No assumption on the (e.g., harmonic) nature of oscillations $\Rightarrow$ 1 *nonlinear* oscillation = 1 mode
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EMD vs. wavelets

1. *similarity* : both achieve a decomposition into "fluctuations" and "trend"

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x(t) &= \sum_k c_k(t) + r_K(t) && \text{(EMD)}\\
&= \sum_k d_k(t) + a_K(t) && \text{(DWT)}\\
\text{with } d_k(t) &= \sum_n \langle x, \psi_{kn}\rangle\, \psi_{kn}(t)\\
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2. *difference* : scales are pre-determined for DWT ($\{\varphi, \psi\}_{kn}(t) = 2^{-k/2}\{\varphi, \psi\}(2^{-k}t - n)$) and adaptive (data-driven) for EMD

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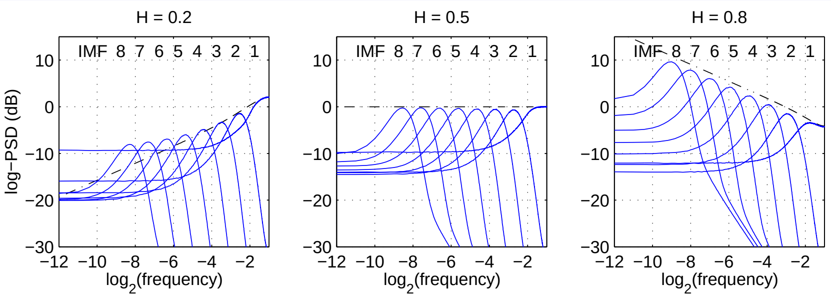
- “spontaneous”, almost dyadic, filterbank organization



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[F. *et al.*, *IEEE Sig. Proc. Lett.*, 2004]

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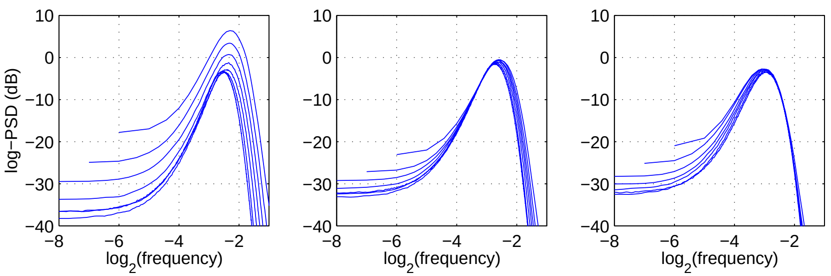
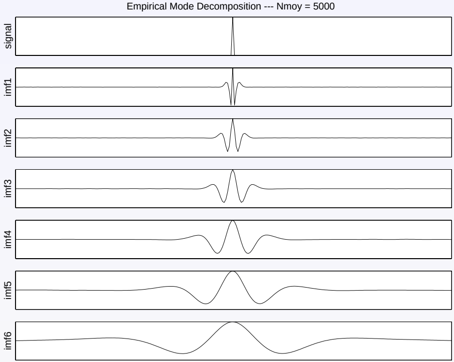


$$\mathcal{S}_{k',H}(f) \approx 2^{(2H-1)(k'-k)} \mathcal{S}_{k,H}\left(2^{k'-k} f\right)$$

[F. *et al.*, *IEEE Sig. Proc. Lett.*, 2004]

Interpretation

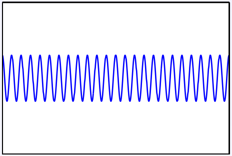
1. *time-domain* : EMD “ìmpulse response”



2. *frequency domain* : 2-tone example

One or two components ?

$$\text{«}\ \cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\ \text{»}$$



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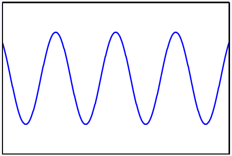
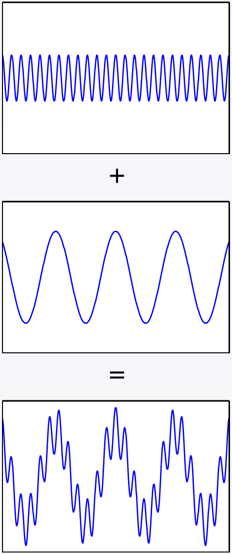


One or two components ?

« $\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$ »



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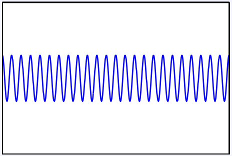

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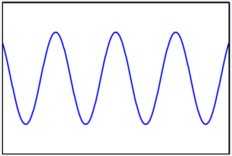

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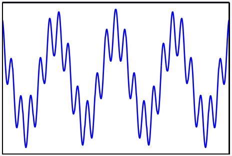

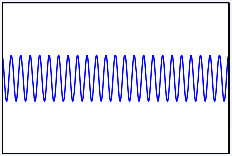

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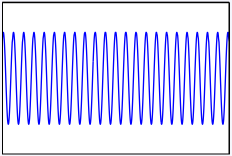

One or two components ?

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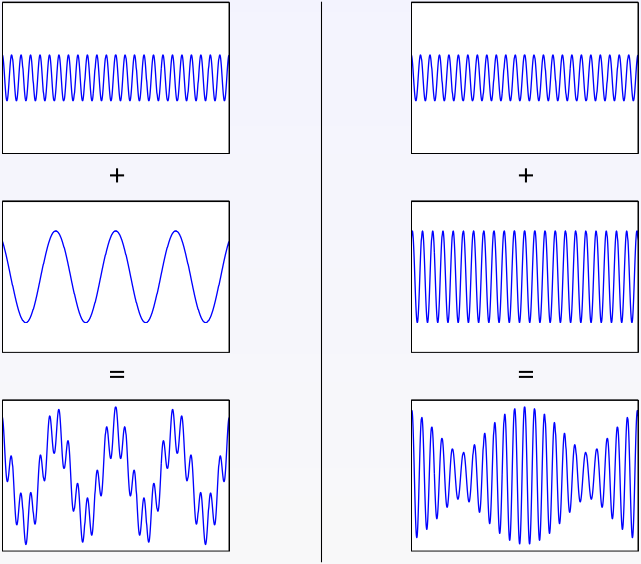

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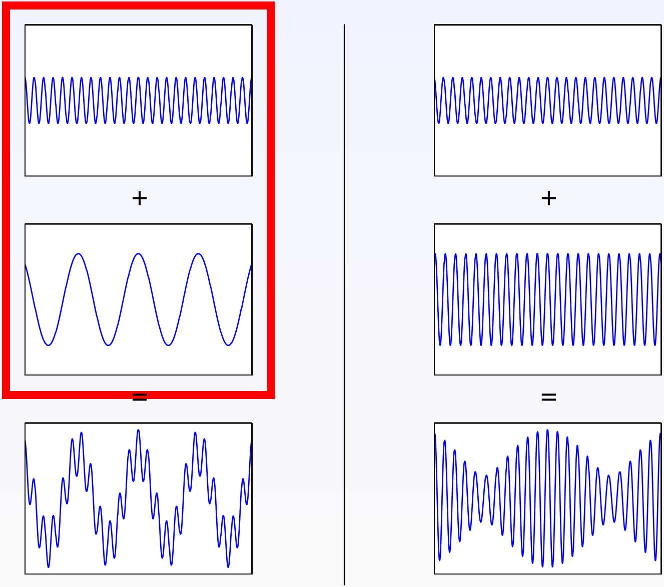

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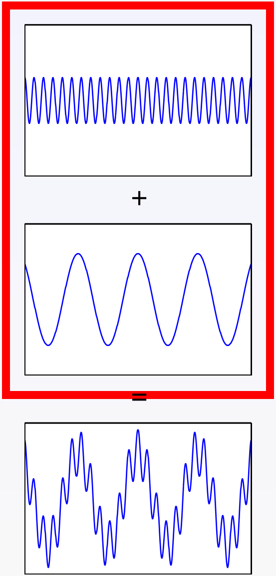

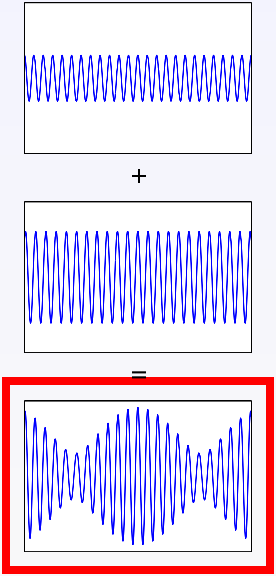

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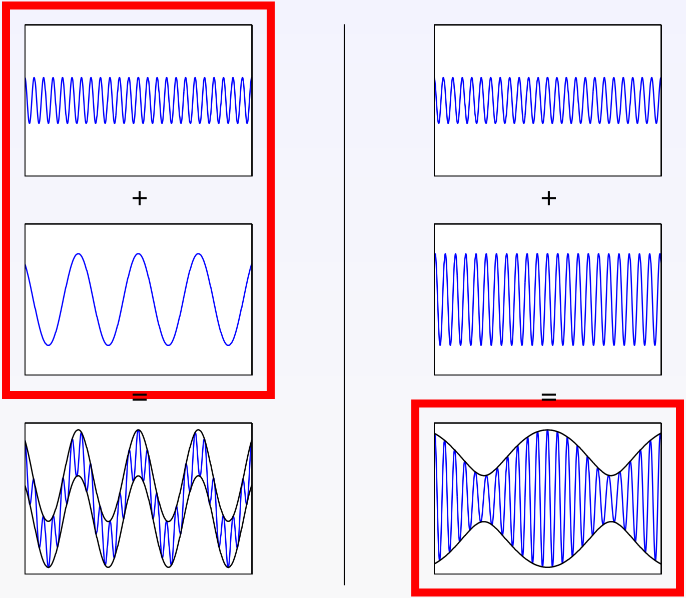

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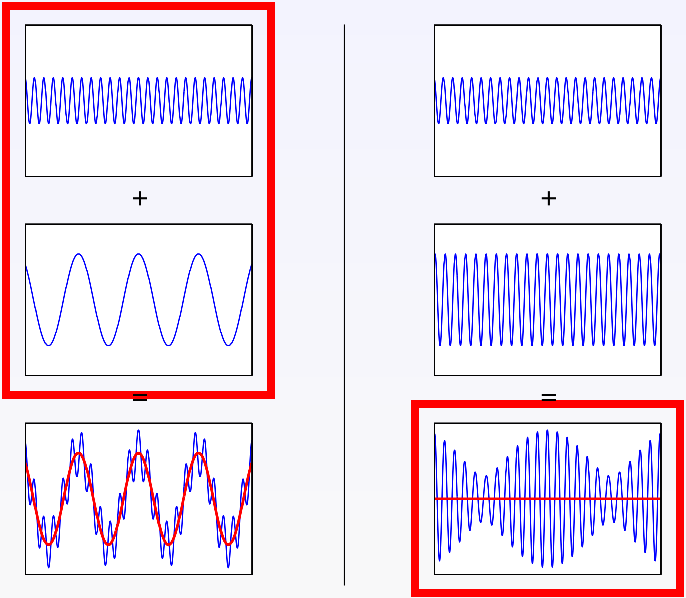

Simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

Analysis of its EMD

- only the first IMF is computed : if separation, it should be equal to the highest frequency component $x_1(t)$
- criterion ($= 0$ if separation) :

$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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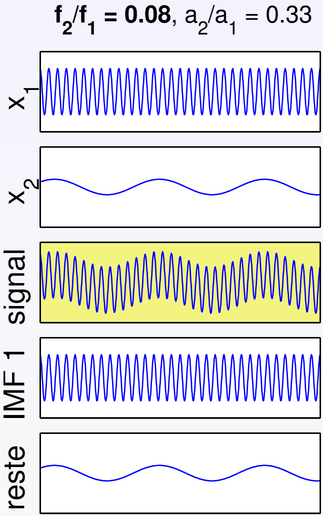
Sum of two tones



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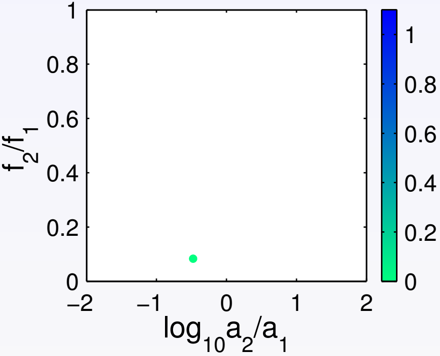
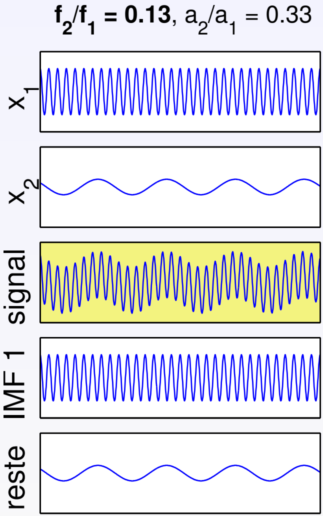


Sum of two tones



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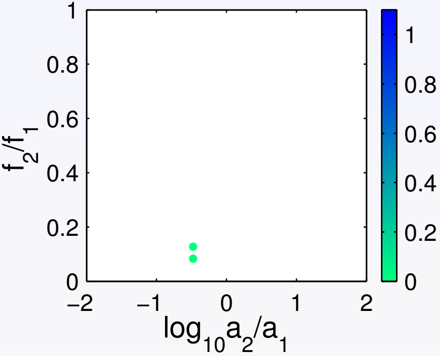
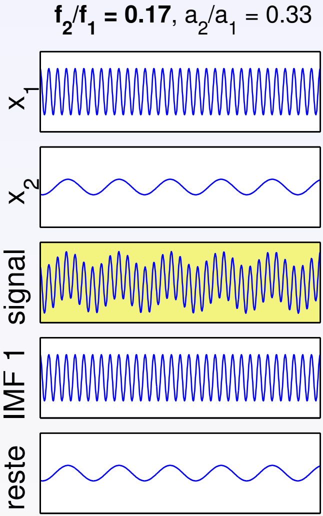


Sum of two tones



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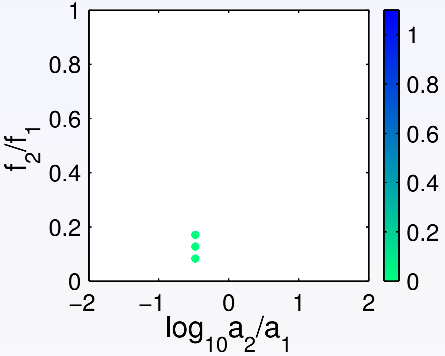
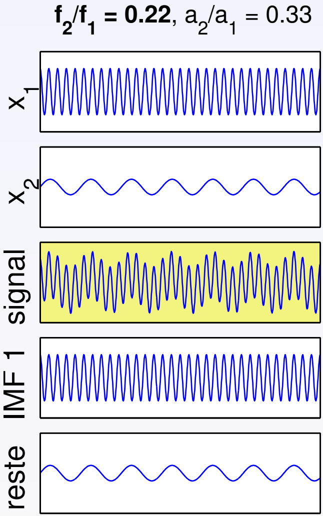


Sum of two tones



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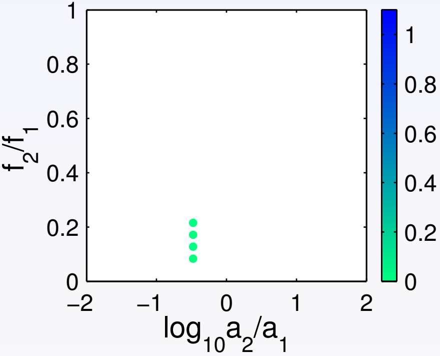
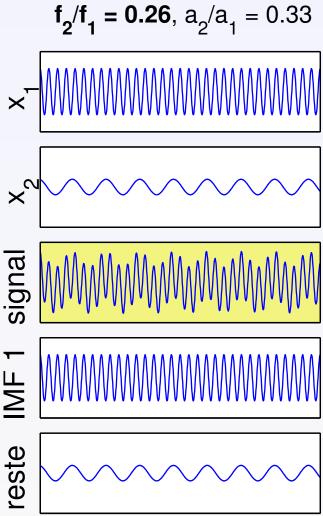


Sum of two tones



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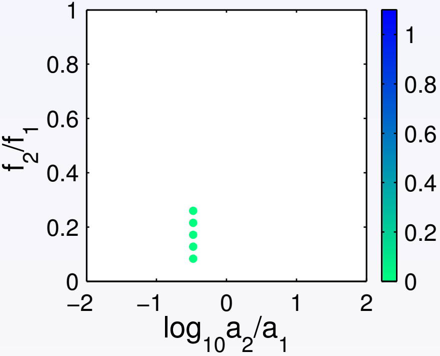
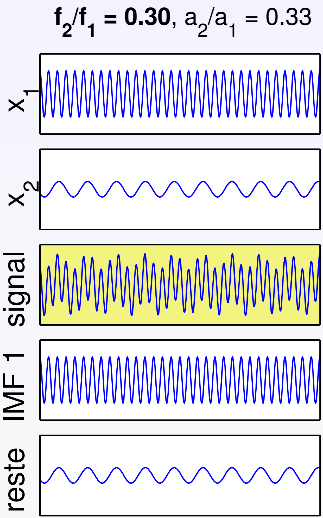


Sum of two tones



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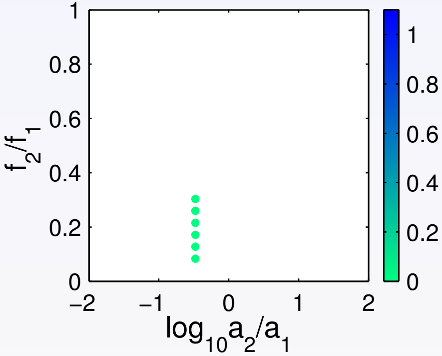
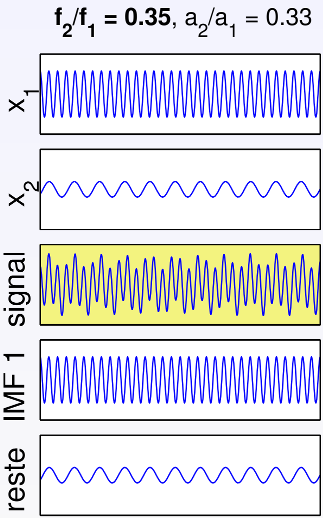


Sum of two tones



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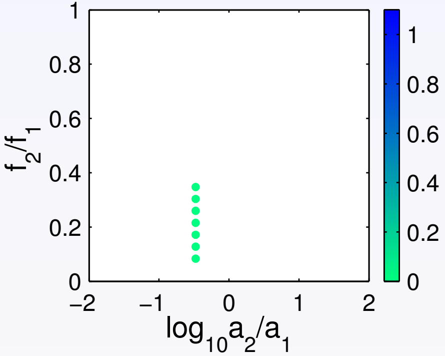

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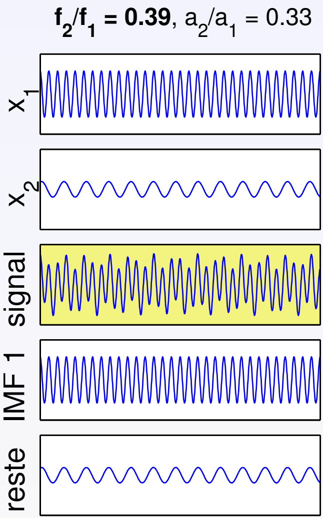

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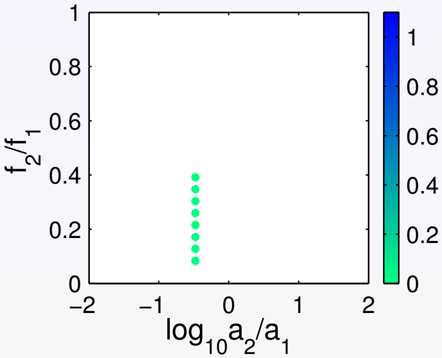

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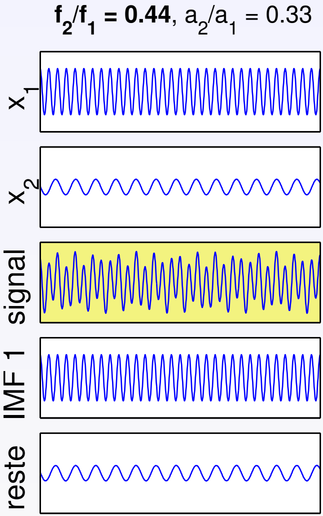

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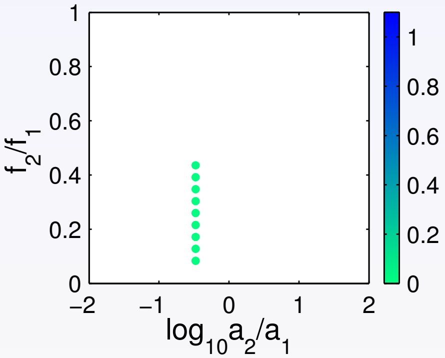

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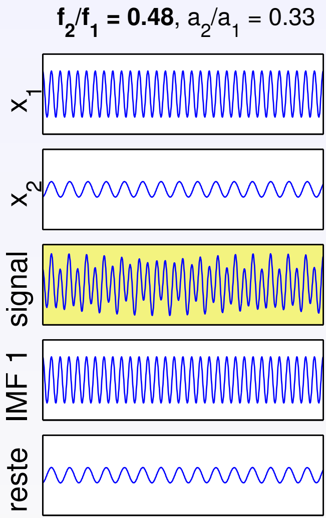

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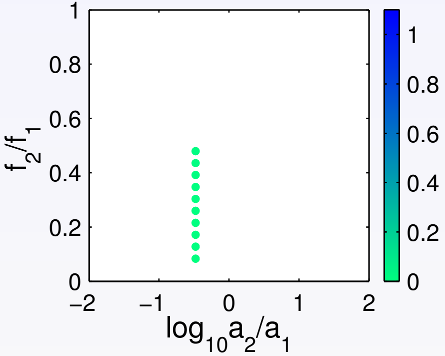
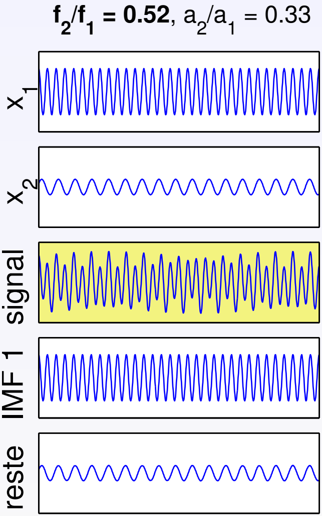


Sum of two tones



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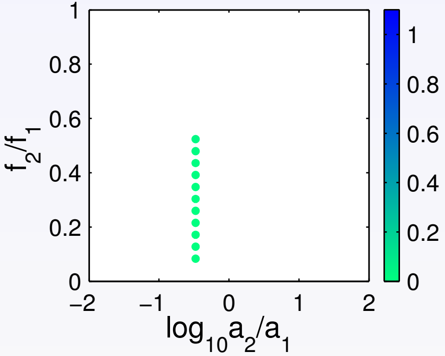

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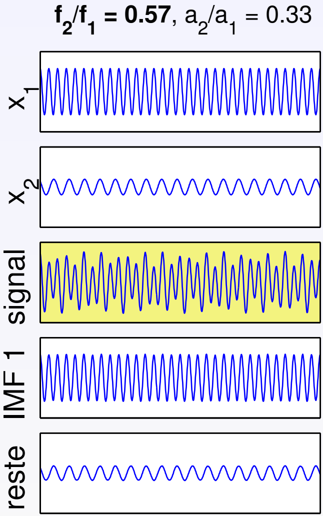

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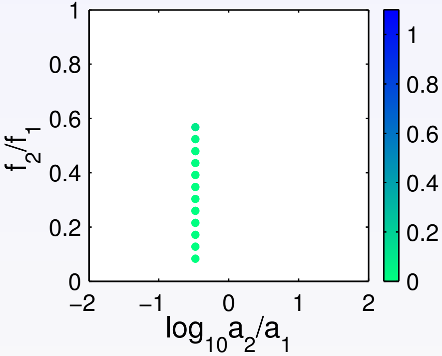

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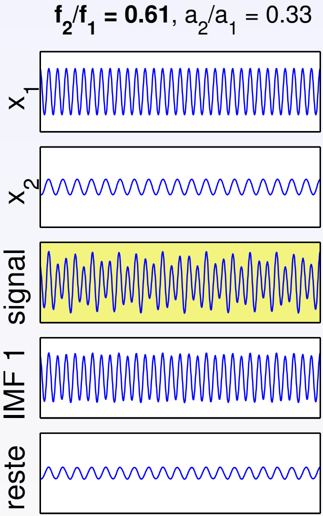

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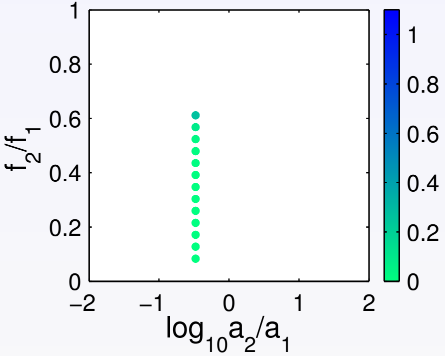

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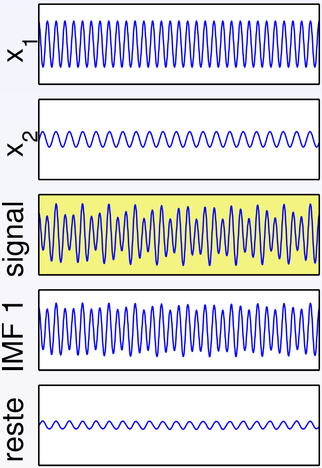

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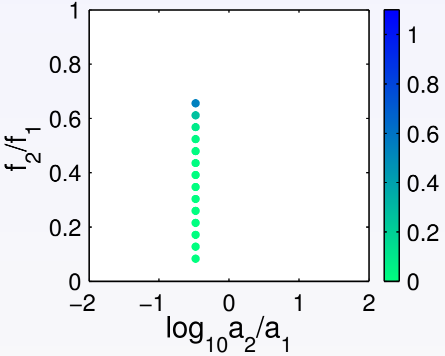

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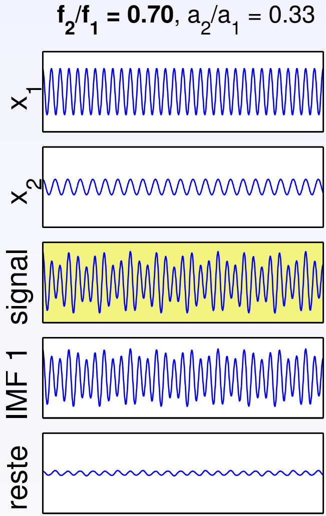

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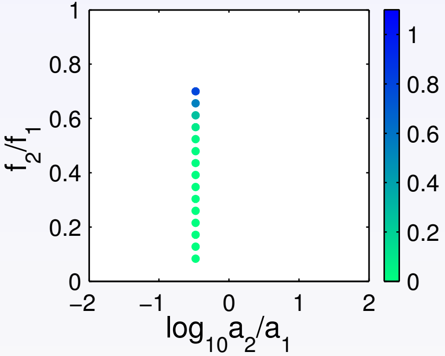

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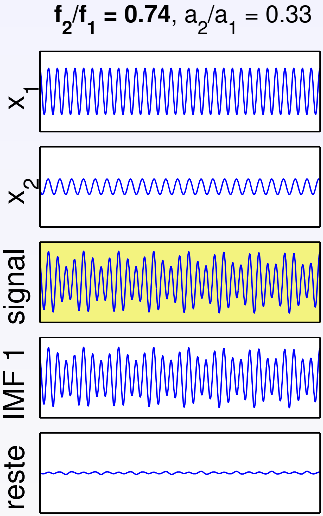

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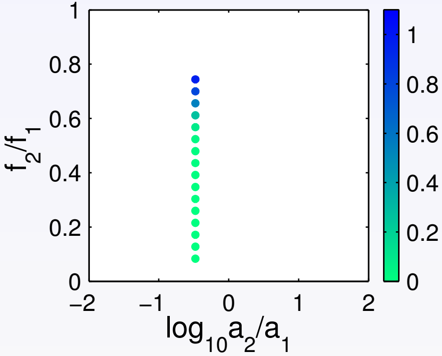

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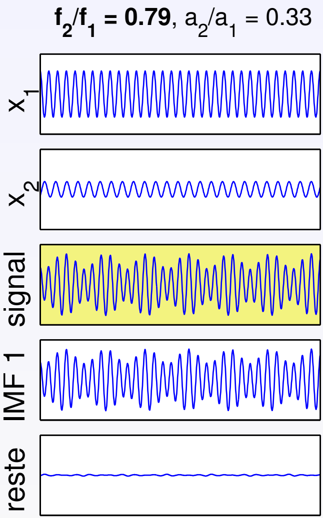

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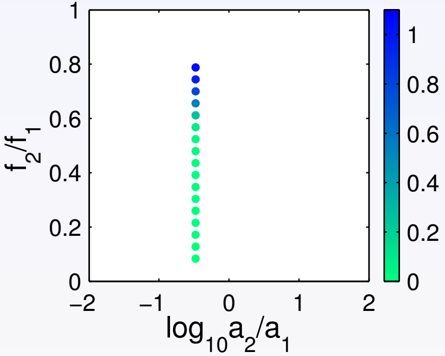
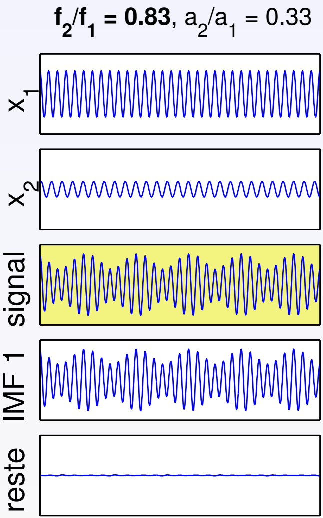
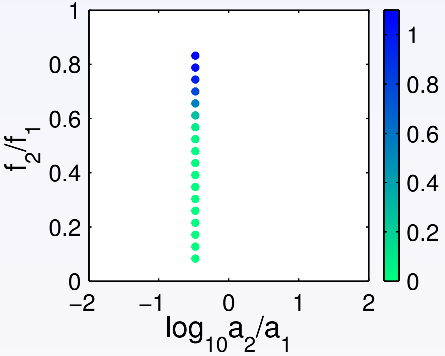


Sum of two tones

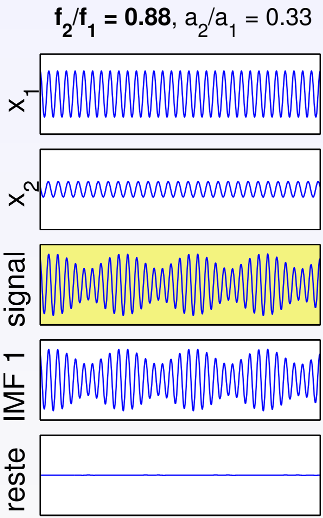


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}$$

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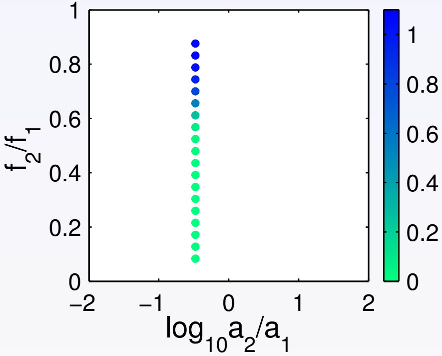

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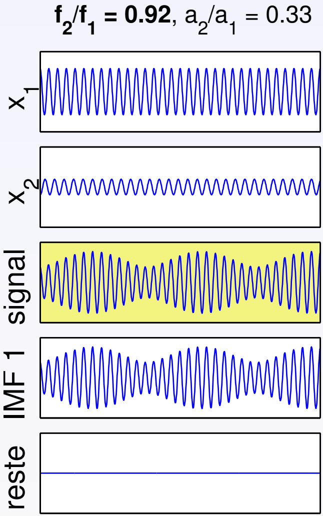

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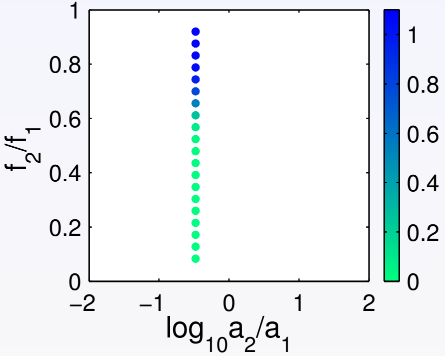
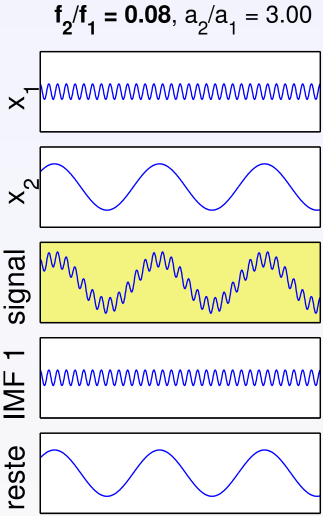


Sum of two tones



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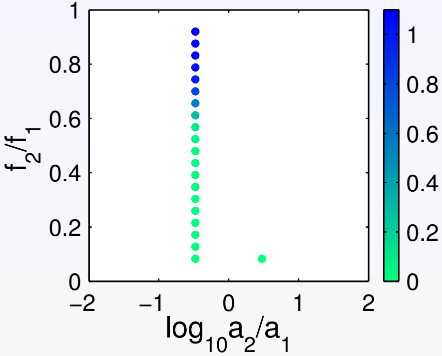

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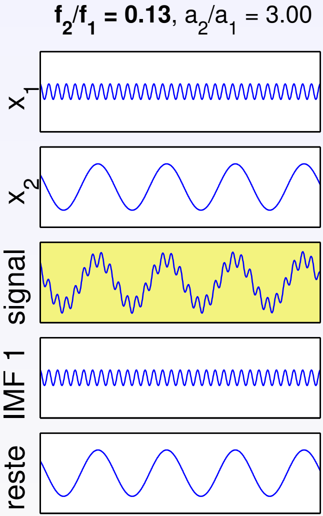

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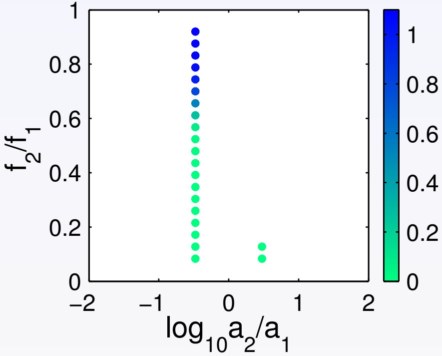
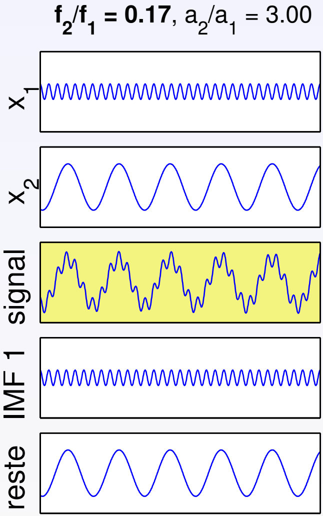


Sum of two tones



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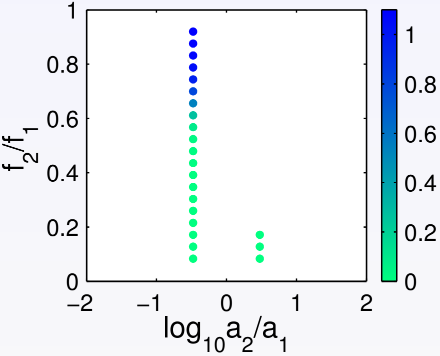
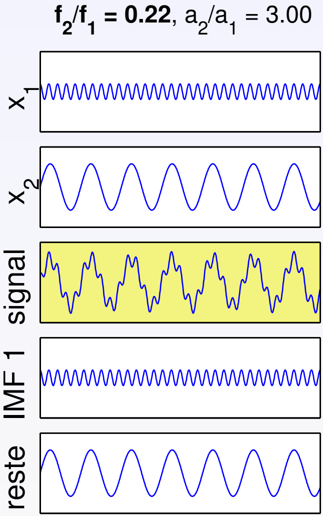


Sum of two tones



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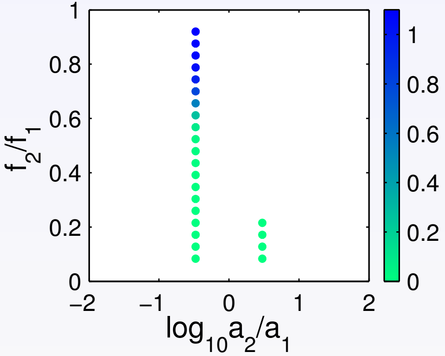
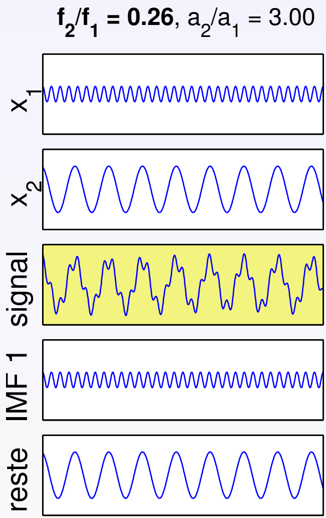


Sum of two tones



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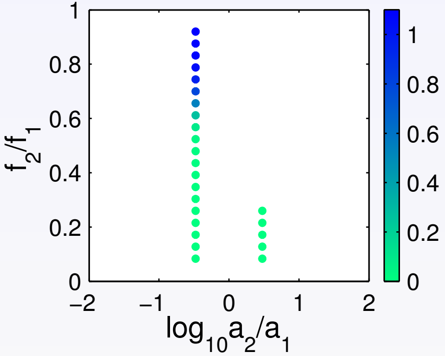
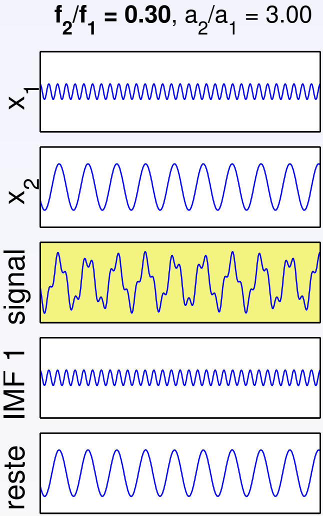


Sum of two tones



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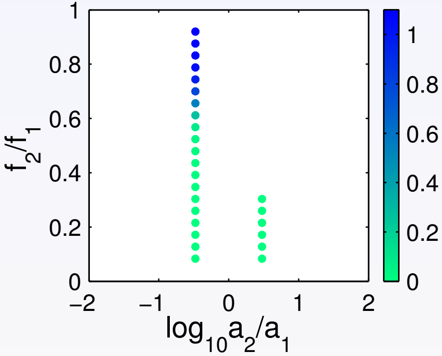
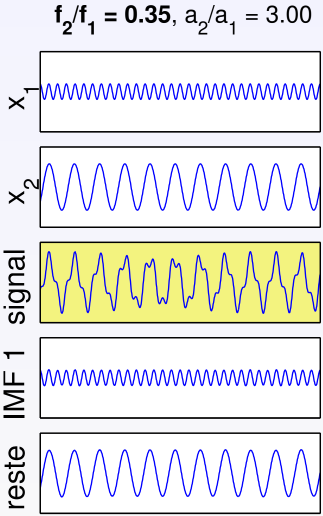


Sum of two tones



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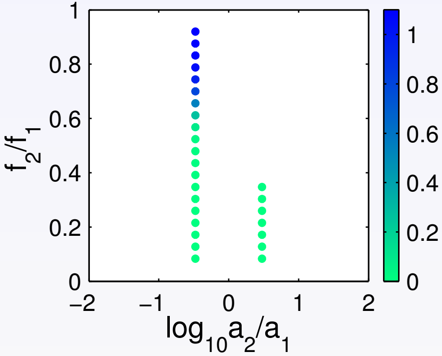

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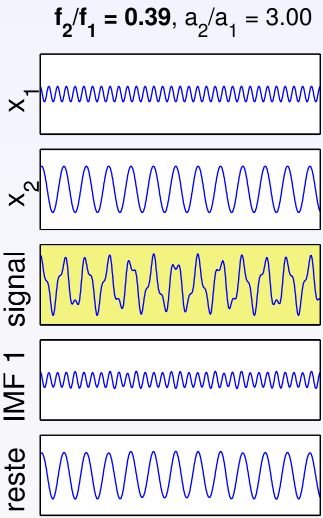

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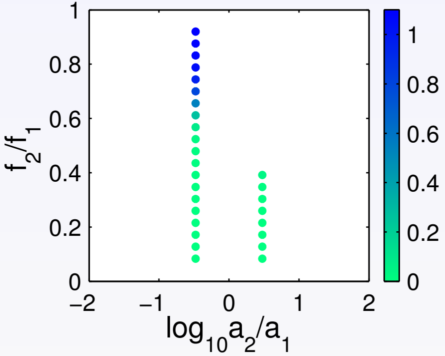
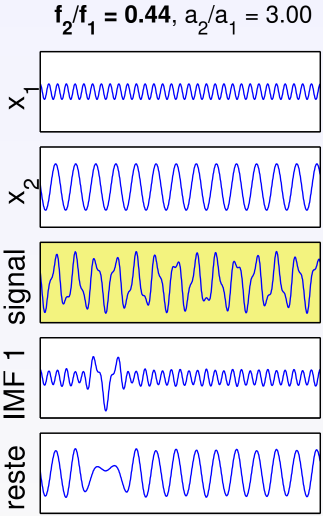


Sum of two tones



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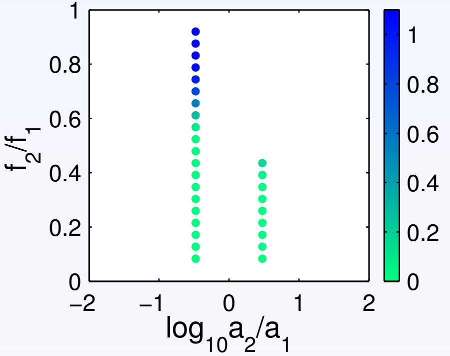

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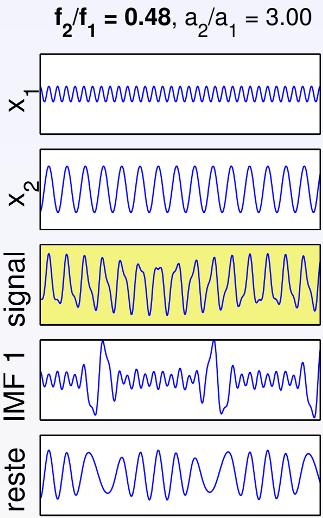

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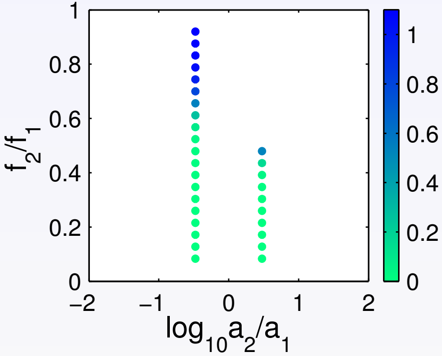

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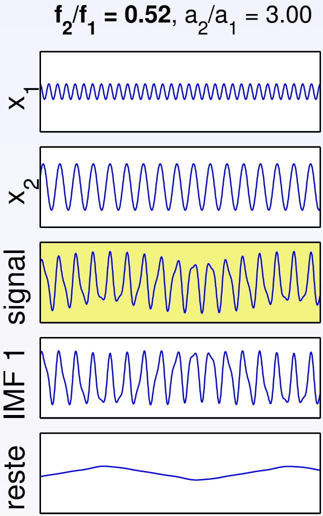

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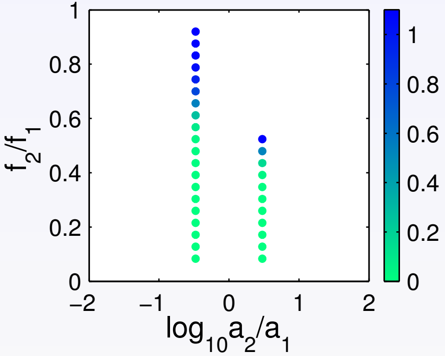

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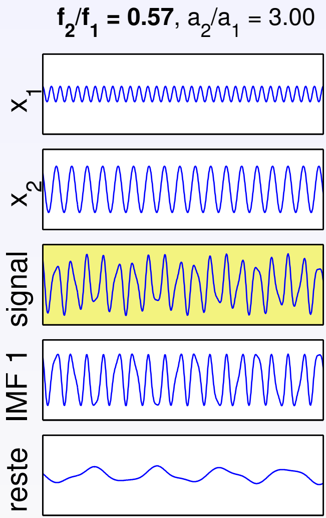

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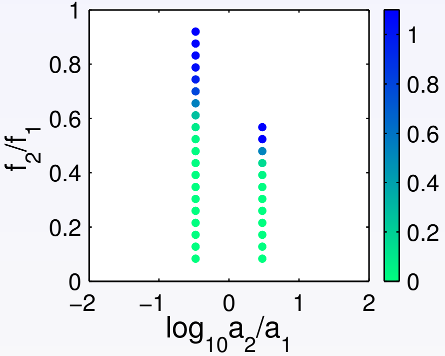

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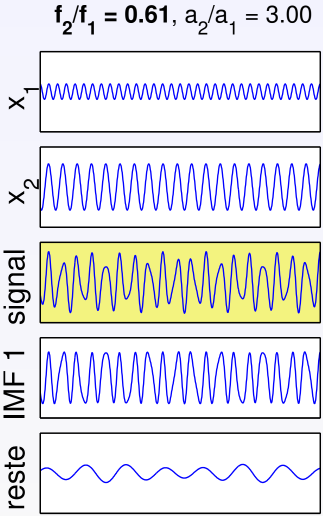

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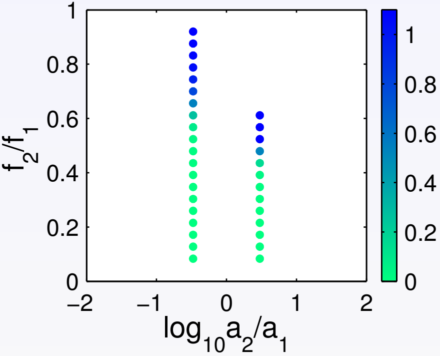

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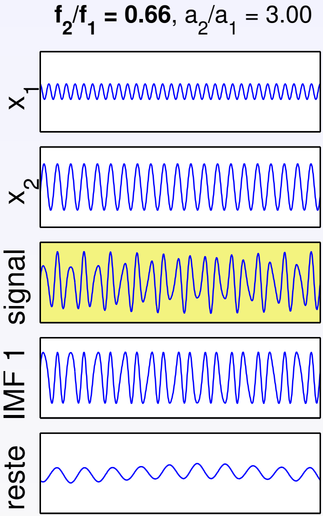

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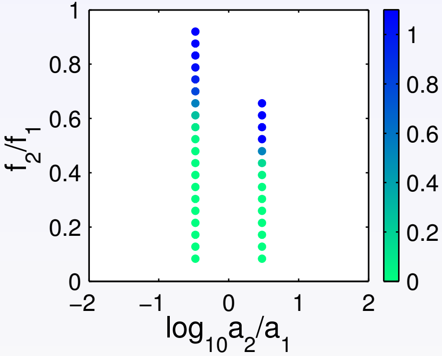

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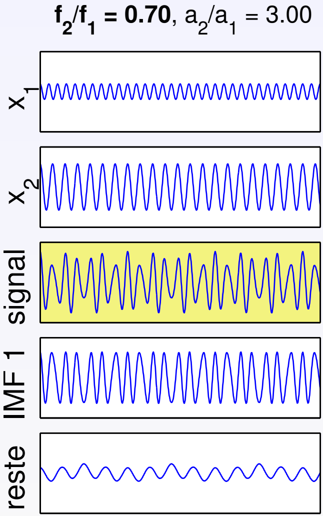

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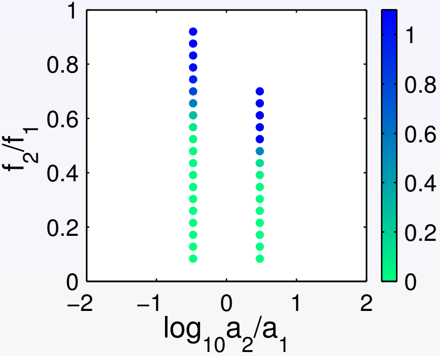

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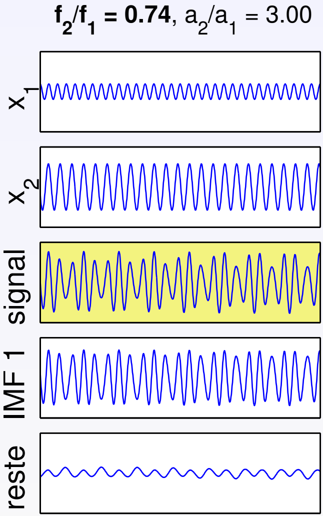

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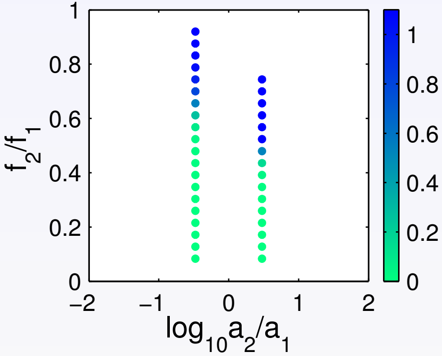

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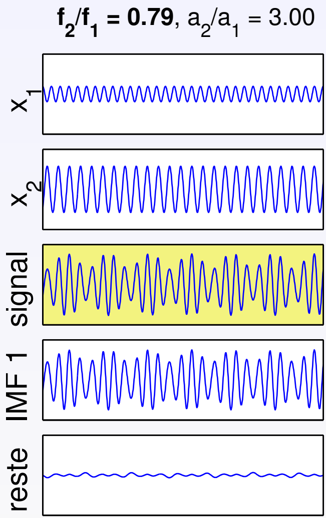

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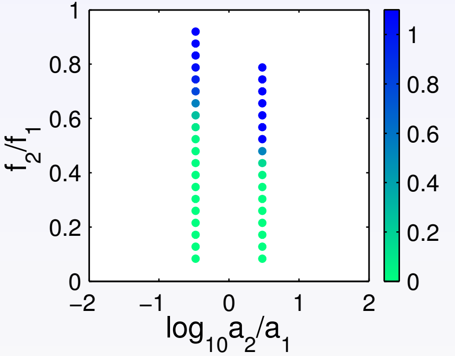

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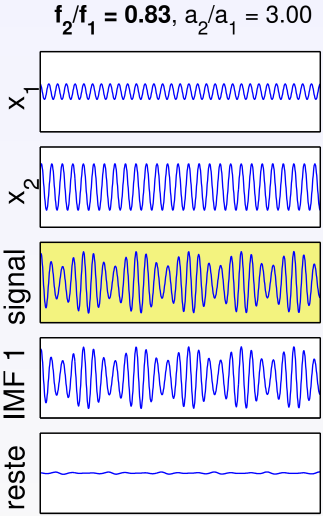

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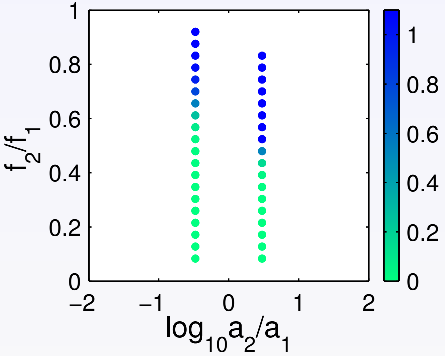

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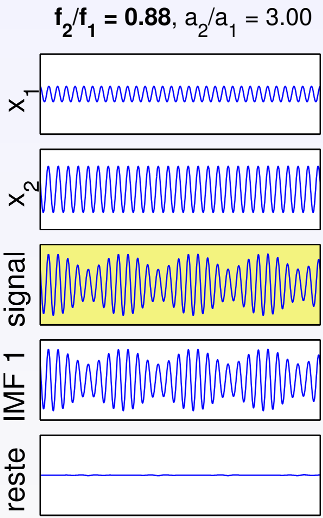

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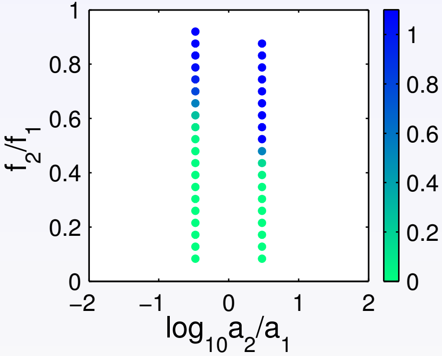

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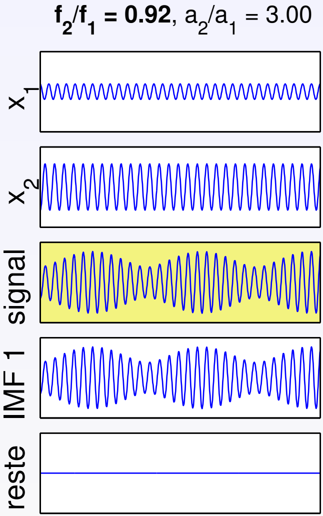

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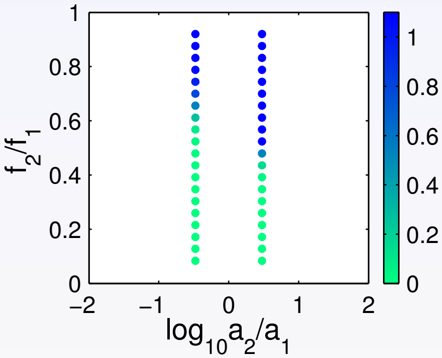

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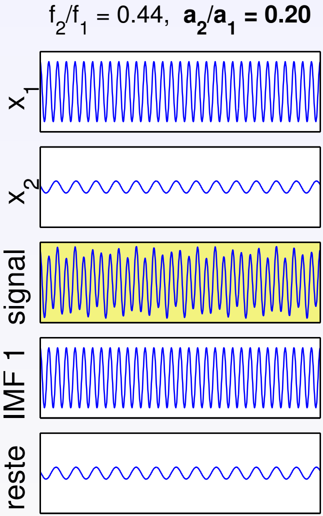

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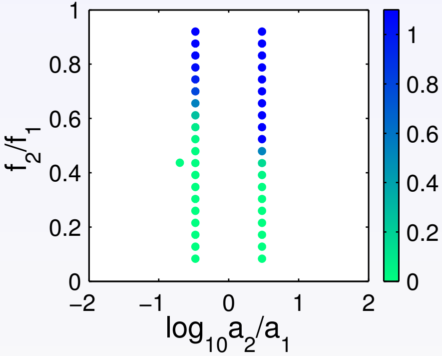

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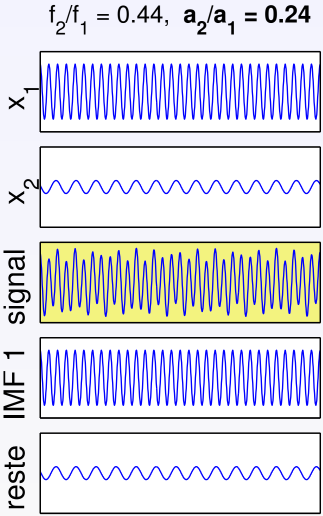

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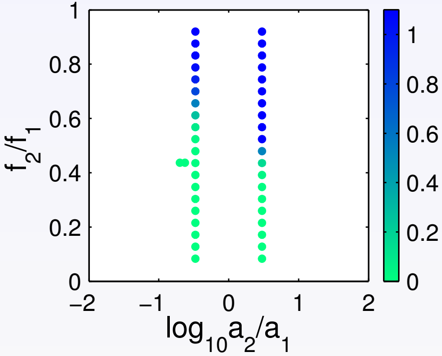

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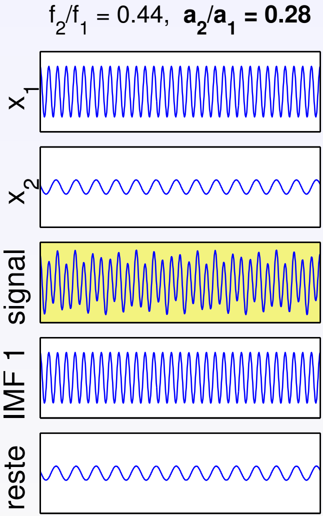

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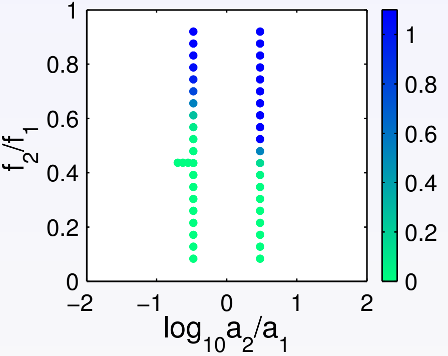
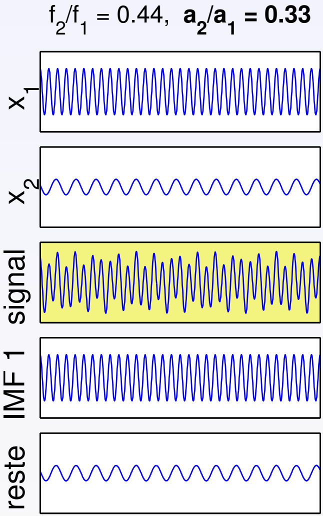


Sum of two tones



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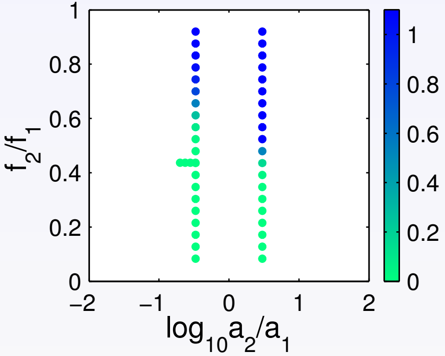
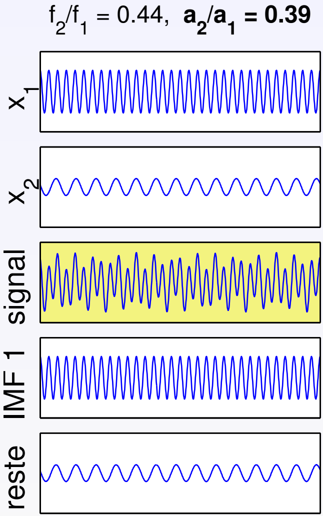


Sum of two tones



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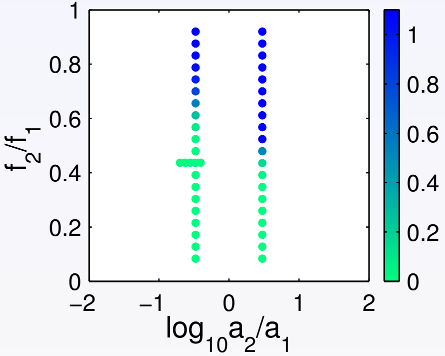

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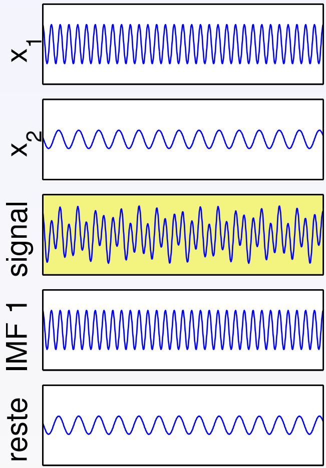

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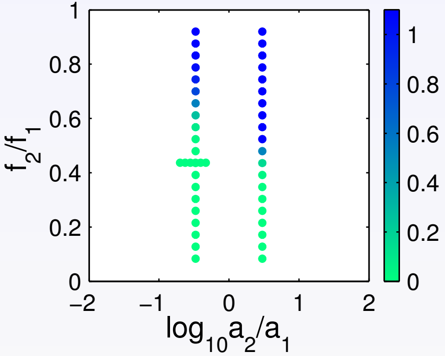

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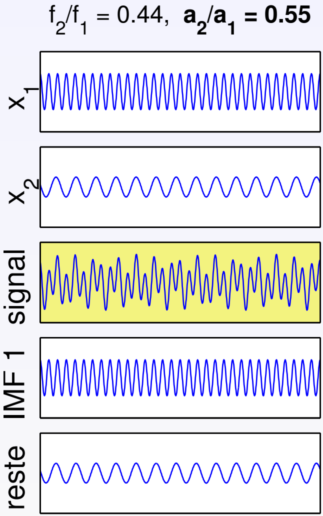

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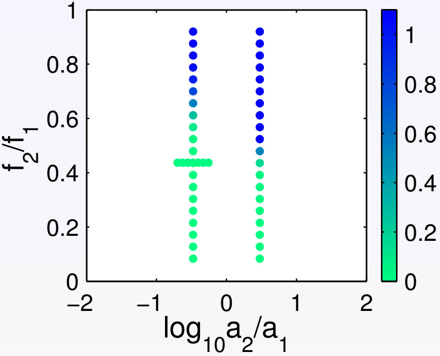

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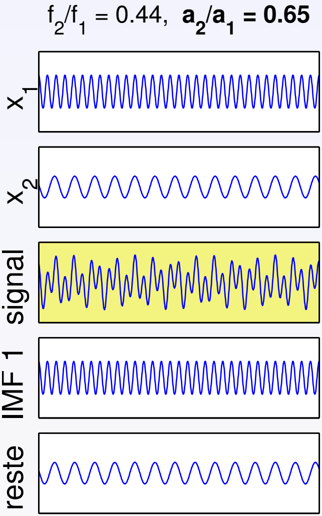

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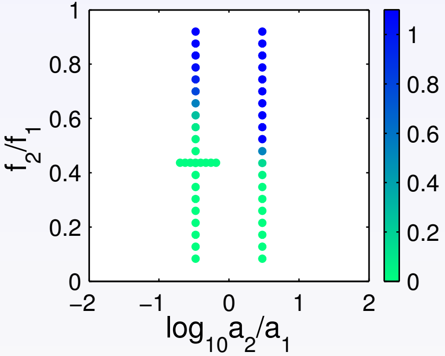

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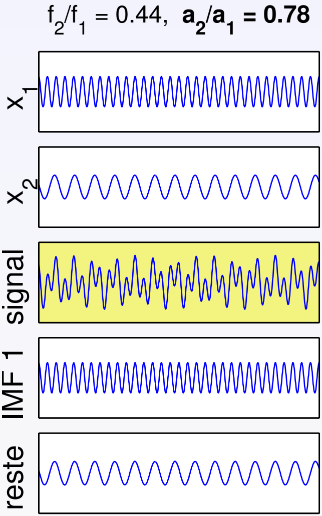

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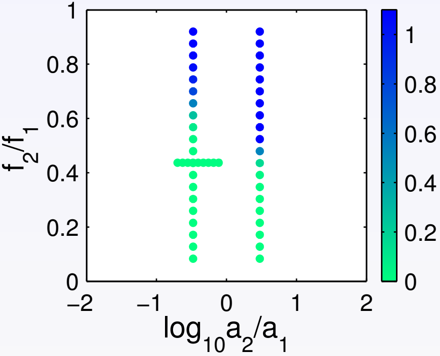

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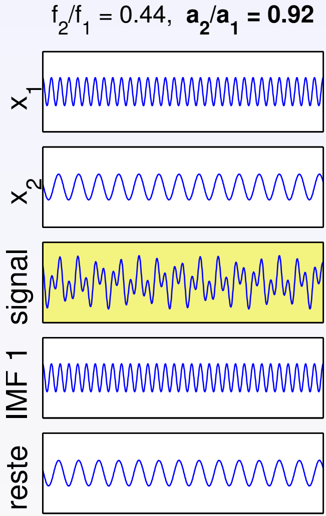

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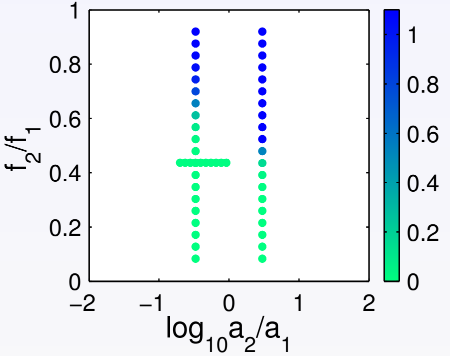

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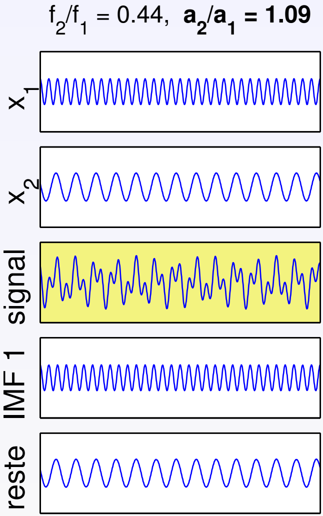

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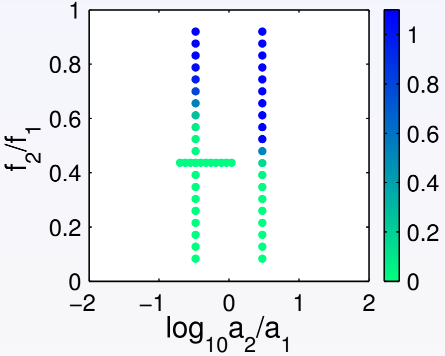

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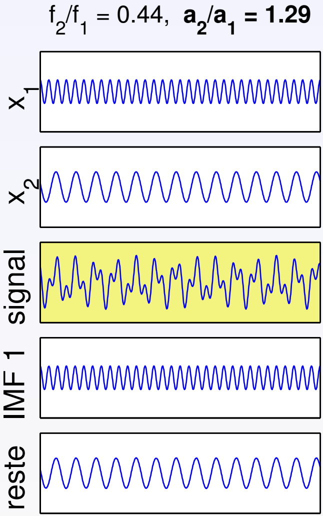

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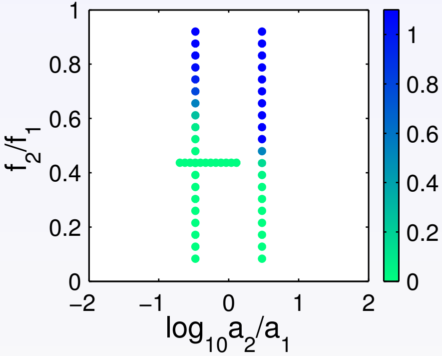

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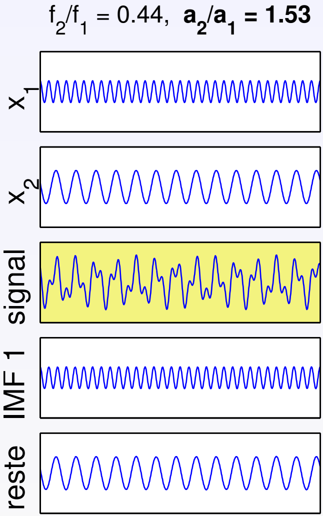

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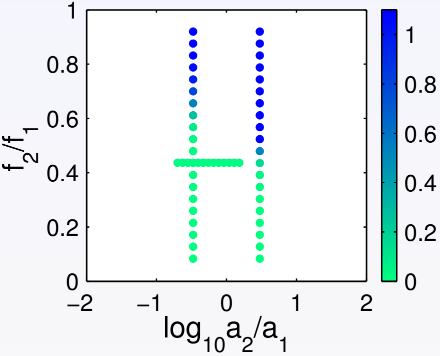

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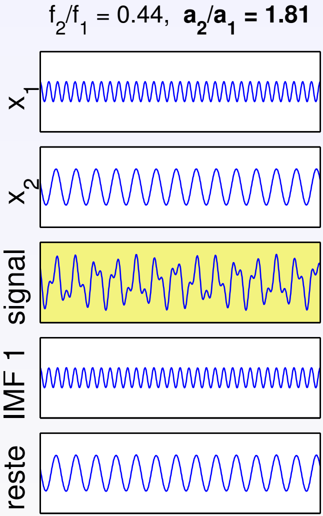

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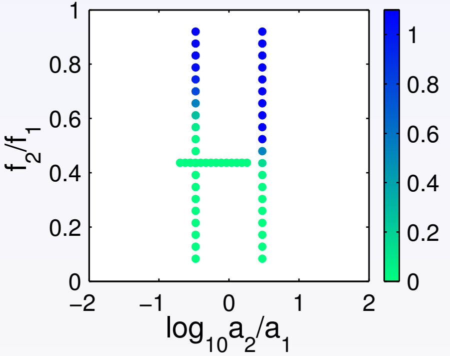

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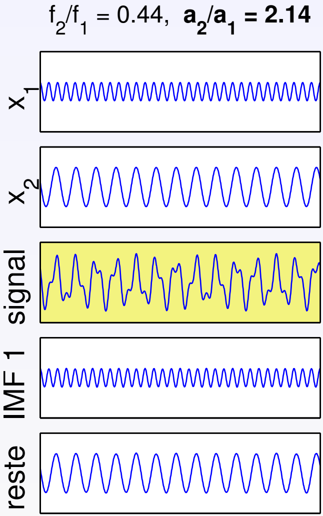

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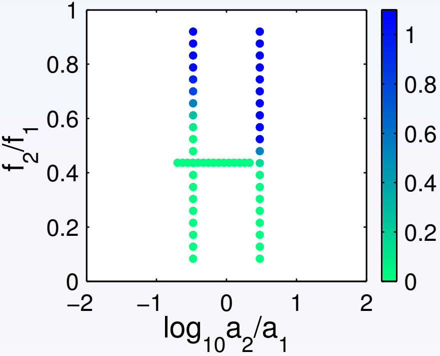

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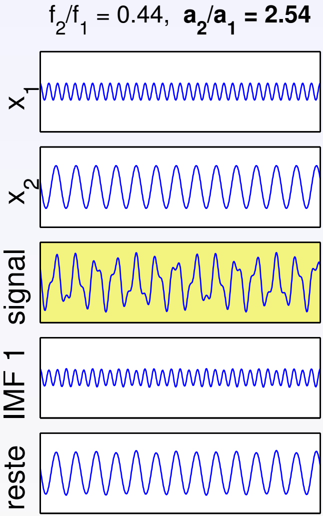

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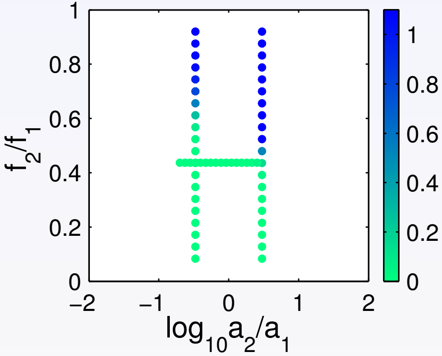

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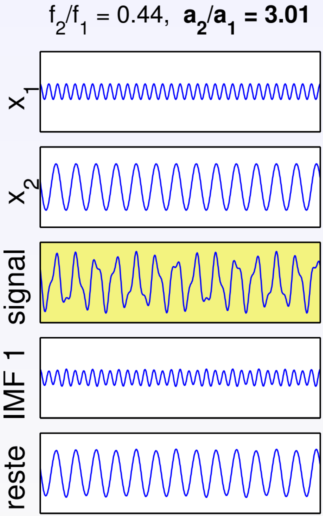

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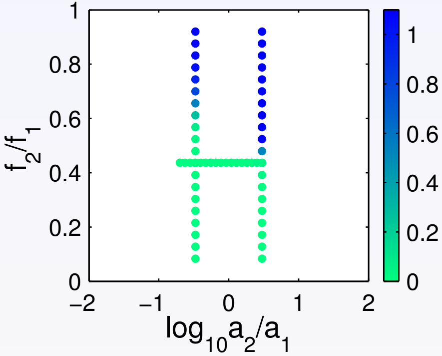

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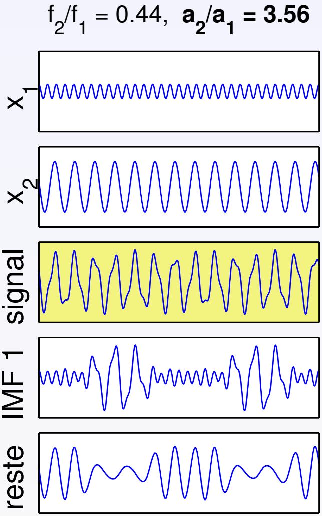

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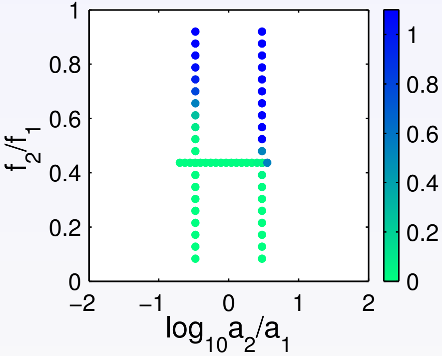

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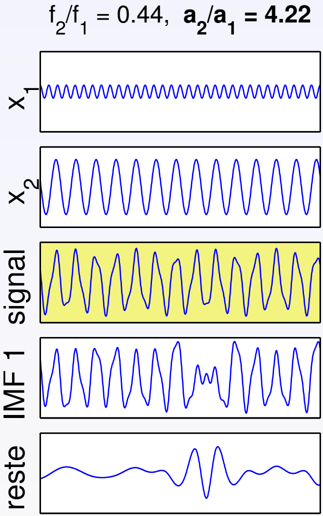

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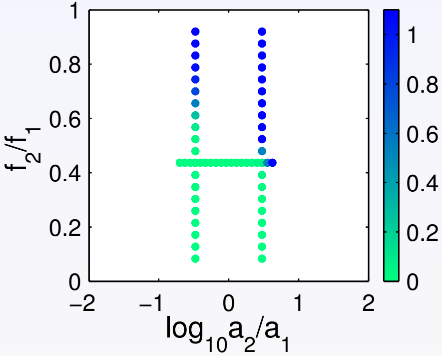

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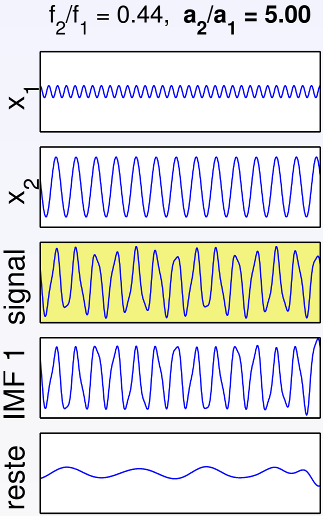

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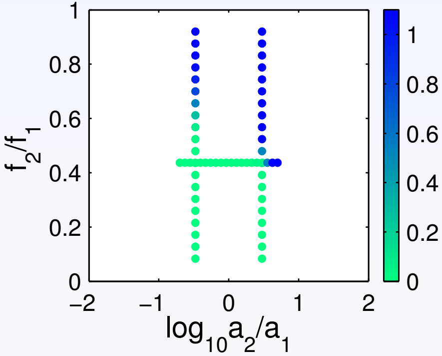

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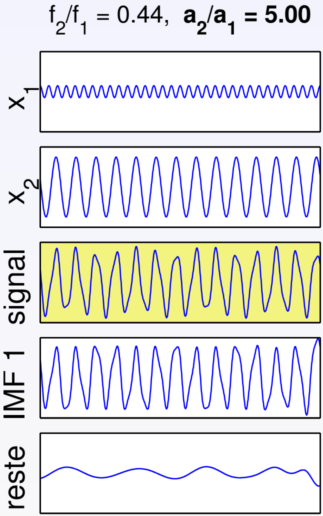

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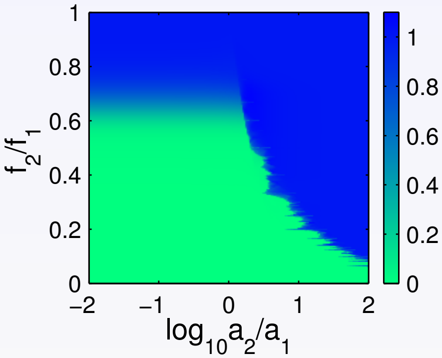

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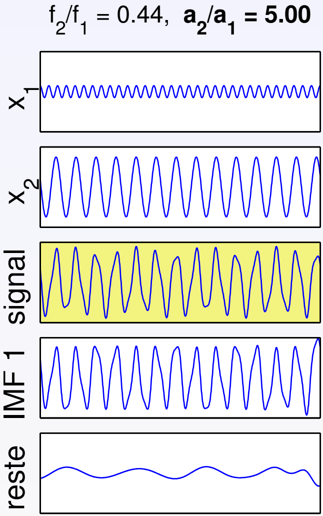

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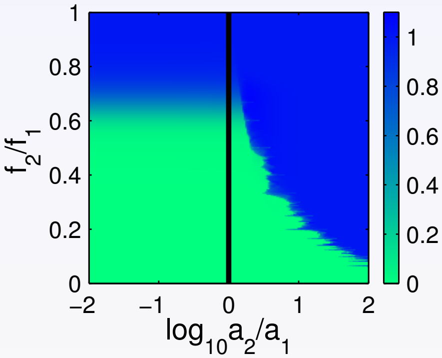

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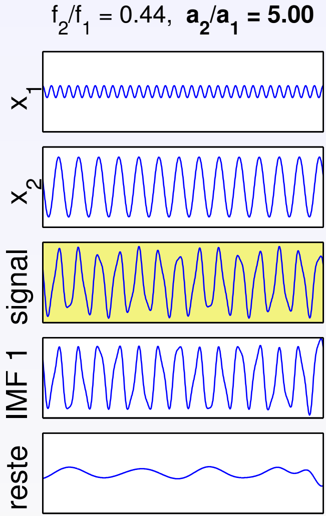

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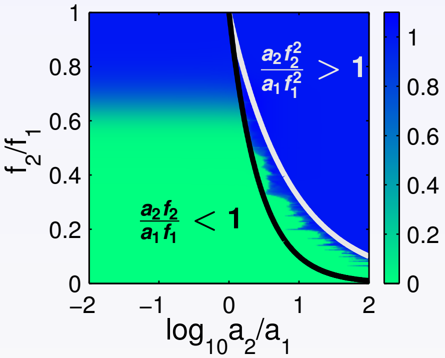

Scaling estimation

1. Multiresolution decomposition of $x(t)$ into scale-dependent contributions $x_k(t)$
2. Evaluation of fluctuation properties (e.g., variance) as a function of scale k
3. Estimation of Hurst exponent from slope α in a log-log plot

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EMD vs. wavelets for scaling estimation

1. wavelet analysis based on log-scale diagrams

$$\log_2 V[k] = \alpha\, k + C,$$

with $V[k] = \operatorname{var} d_k[n]$, estimated by $\hat{V}[k] = \overline{d_k^2[n]}$, as a function of the fixed scale k

2. EMD analysis based on “modegrams” ($d_k[n] \to c_k[n]$), with variance a function of either the IMF index k or the natural scale given by the mean period $T[k]$, estimated as

$$\hat{T}[k] := \frac{\text{distance between the first and the last zero-crossing}}{\text{number of zero-crossings} - 1}$$

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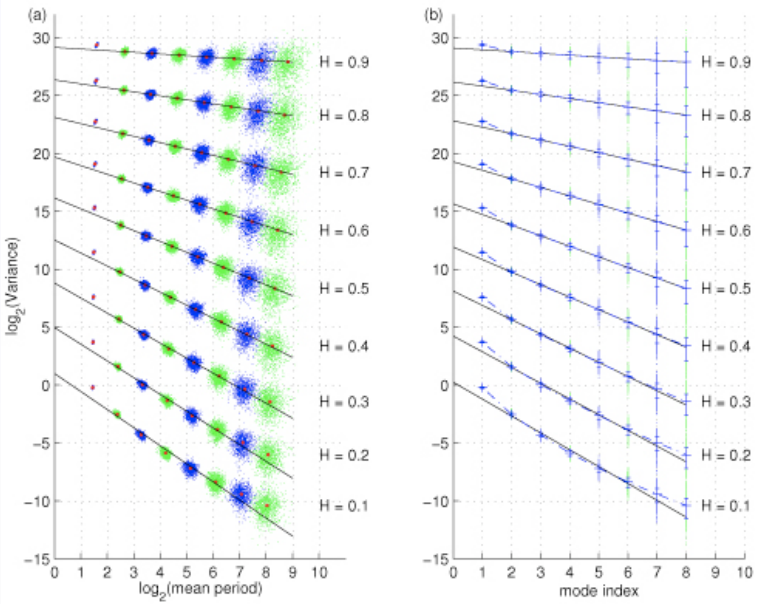
Modegrams of fGn



Dimension estimation

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2. Evaluation of the length of the corresponding graph

$$\mathcal{L}(a_k) = \int \sqrt{1 + \dot{a}_k^2(t)}\, dt$$

3. Estimation of (regularization) dimension D_x from slope in a log-log plot (small-scale limit)

$$\boxed{\text{fBm} : D_x = 2 - H}$$

[Lévy-Vehel & Roueff, 1998]

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EMD vs. wavelets for regularization dimension estimation

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2. EMD

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Experimental comparison

- Synthetic fBm
  - circulant matrix method of [Wood & Chan, 1994]
  - $H = 0.1, 0.2, \ldots, 0.9$
  - 10,000 realizations of 1024 data points
- comparison of 6 estimators
  - 3 variance-based : DWT, wavelet leaders and EMD
  - 3 length-based : smoothed graph, DWT and EMD
- Weighted linear regressions in log-log plots
- Performance measured by mean-square error and distribution of estimates

[Gonçalvès *et al.*, 2007]

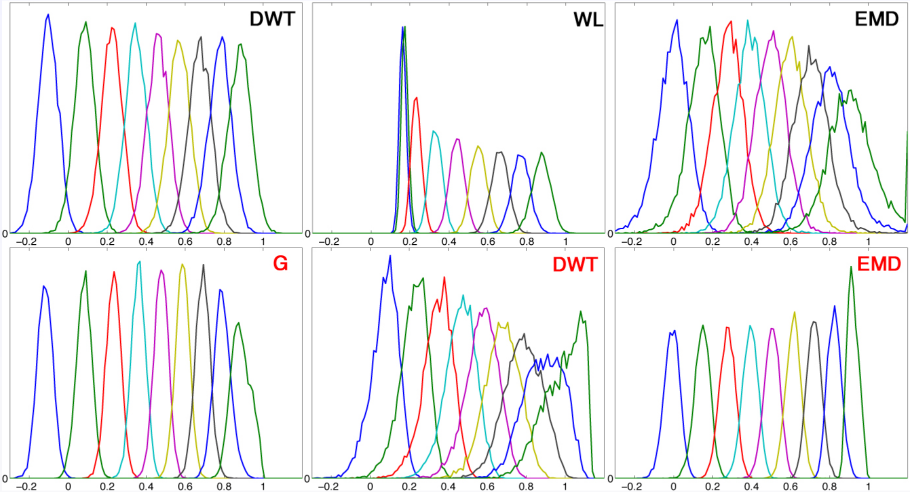
Mean Square Error

$$\mathrm{MSE} = (\mathbb{E}\,\hat{H} - H)^2 + \mathrm{var}\,\hat{H}$$

for *variance-based* (**DWT**, wavelet leaders (**WL**) and **EMD**) and *length-based* (smoothed graph (G), DWT and EMD) estimators

	DWT	**WL**	**EMD**	G	DWT	EMD
$H = 0.1$	.0439	**.0044**	.0194	.0497	*.0057*	.0135
$H = 0.2$	.0166	**.0010**	.0099	.0156	.0058	*.0044*
$H = 0.3$	.0090	*.0051*	.0074	.0062	.0079	**.0023**
$H = 0.4$	.0060	.0059	.0083	*.0030*	.0102	**.0016**
$H = 0.5$	.0048	.0049	.0069	*.0019*	.0123	**.0016**
$H = 0.6$	.0044	.0041	.0076	**.0015**	.0139	*.0017*
$H = 0.7$	.0042	.0034	.0088	**.0016**	.0163	*.0018*
$H = 0.8$	.0040	.0031	.0117	*.0020*	.0181	**.0019**
$H = 0.9$	.0041	.0030	.0888	*.0027*	.0175	**.0012**

Distribution of estimates



Facts on EMD-based estimations

1. compare favorably with more “classical” estimations
2. length better than variance
3. better than wavelets on short data records
4. adapt automatically to “natural” scales $\Rightarrow$ easier identification of “inertial ” range and assessment of scaling
5. robustness to superimposed periodic components and possible extensions [Huang, Schmitt *et al.*, 2008-2009]

Concluding remarks

- “Classical” EMD
  - data-driven multiresolution decomposition $\Rightarrow$ adptation to natural scales (scaling assessment and estimation)
  - empirical approach $\Rightarrow$ no firm theoretical framework
- Two possible ways out
  - work towards a comprehensive theory of current EMD (e.g., PDE formulations)
  - consider more tractable variations keeping the flavour of EMD (e.g., “synchrosqueezing” of Daubechies *et al.*)

EMD and multifractals

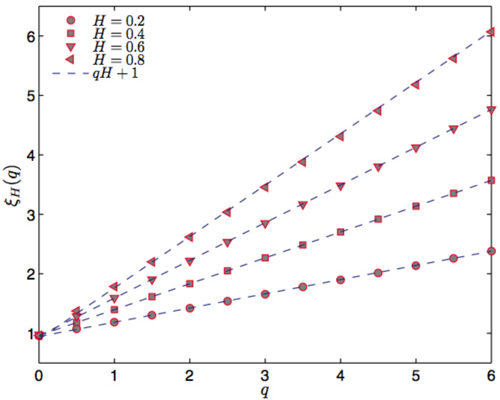


Fig. 3: Scaling exponents $\xi(q)$ for fractional Brownian motion simulations with $H = 0.2$, 0.4, 0.6 and 0.8, respectively.

[Huang, Schmitt *et al.*, 2008]

EMD and multifractals

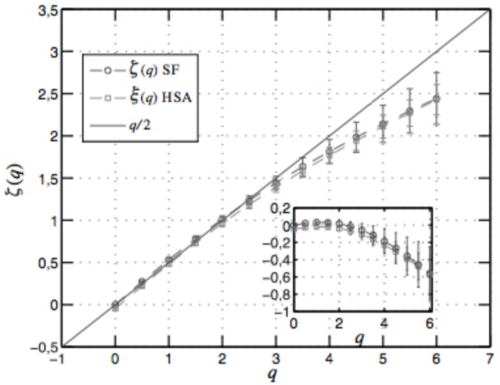


Figure 7. Comparaison de l'estimation de l'exposant multifractal à l'aide des fonctions de structure et à l'aide de l'analyse spectrale de Hilbert d'ordre quelconque pour des simulations multifractales lognormales avec $\mu = 0.05$.

[Huang, Schmitt *et al.*, 2009]

Turbulence data

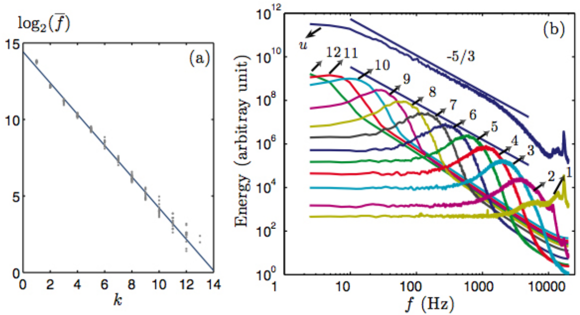


Fig. 2: (a) Mean frequency *vs.* mode number for the turbulent velocity time series. There is an exponential decrease with a slope very close to 1. This indicates that EMD acts as a filter bank which is almost dyadic. (b) Fourier spectrum of each mode (from 1 to 12) showing that they are narrow-banded. The slope of the reference line is $-5/3$ corresponding to the inertial-range Kolmogorov spectrum.

[Huang, Schmitt *et al.*, 2008]

Turbulence data

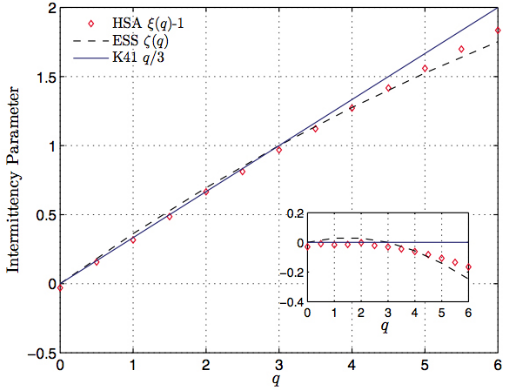


Fig. 8: Comparison of the scaling exponents $\xi(q) - 1$ (diamond) with the classical $\zeta(q)$ obtained from structure functions analysis with the ESS method (dashed line) and K41 $q/3$ (solid line). The insert shows the departure from the K41 law.

[Huang, Schmitt *et al.*, 2008]