Linear Multifractional Stable Motion : wavelet methods and sample path properties

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Outline

The Linear Multifractional Stable Motion

Definition 1.1 (Stoev & Taqqu'04)

Let $1 < \alpha \leq 2$ *and* $H(\cdot)$ *be a functional parameter with values in* $[a, b] \subset (1/\alpha, 1)$. *The Linear Multifractional Stable Motion (LMSM) process* $Y(t)$ *can be expressed by*

$$Y(t) = X(t, H(t)), \tag{1}$$

$X = \{X(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ *being the* $\mathcal{S}t\alpha\mathcal{S}$ *field defined for all* (u, v) *as the stochastic integral:*

$$X(u, v) = \int_{\mathbb{R}} \left\{(u - s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}\right\} Z_\alpha(ds). \tag{2}$$

Stoev and Taqqu's results : Continuity

$\mathcal{K}$ denotes a fixed compact interval.

Theorem 1.1 (Stoev and Taqqu '05)

Let $\alpha \in (1,2)$ and $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be a LMSM whose parameter $H(\cdot)$ satisfies for all $t^{'}, t^{''} \in \mathcal{K}$,

$$\left| H(t^{'}) - H(t^{''}) \right| \leq c|t^{'} - t^{''}|^{\rho} \text{ with } \rho > 1/\alpha, \quad (3)$$

$c > 0$ being a constant which does not depend on $t^{'}, t^{''}$. Then, with probability 1 (w.p.1) the process Y has continuous paths on $\mathcal{K}$.

Conjecture : The continuity of the paths of LMSM holds as long as $H(\cdot)$ is continuous. By using Daubechies wavelets, we will prove that this Stoev and Taqqu's conjecture is true.

- The conjecture has already been solved in the Gaussian case i.e. $\alpha = 2$ (Ayache and Taqqu '05).
- Our wavelet method also allow to improve some Stoev and Taqqu's results concerning the uniform Hölder regularity of LMSM paths.

Recall that:

(a) Hölder Space : for every $\gamma \in [0,1]$, the space of real-valued γ-Hölder functions on the interval $\mathcal{K}$, is the Banach space

$$\mathcal{C}^{\gamma}(\mathcal{K}, \mathbb{R}) := \{f : \mathcal{K} \to \mathbb{R} : \sigma_{\gamma}(f) < \infty\}, \tag{4}$$

where $\sigma_{\gamma}(f) := \sup_{x \in \mathcal{K}} |f(x)| + \sup_{x,y \in \mathcal{K}} \frac{|f(x)-f(y)|}{|x-y|^{\gamma}}$ is the natural norm;

(b) The (critical) Hölder exponent of a continuous and non-differentiable function g over $\mathcal{K}$ is defined as

$$\beta_g(\mathcal{K}) := \sup\{\gamma \ : \ g \in \mathcal{C}^{\gamma}(\mathcal{K}, \mathbb{R})\}.$$

Theorem 1.2 (Stoev and Taqqu '05)

When $H(\cdot)$ *belongs to the Hölder space* $\mathcal{C}^{\beta}(\mathcal{K}, \mathbb{R})$ *with* $\beta > H^* := \max_{t \in \mathcal{K}} H(t)$ *then*

$$H_* - 1/\alpha \leq \beta_Y(\mathcal{K}) \leq H_*,$$

where $H_* := \min_{t \in \mathcal{K}} H(t)$*.*

Goal : We will give a sharp modulus of continuity of the paths of Y and consequently prove that almost surely (a.s.) $\beta_Y(\mathcal{K}) = H_* - 1/\alpha$.

Outline

We denote by ψ be a **compactly** supported $\mathcal{C}^3$ Daubechies mother wavelet.

Definition 2.1

For any $(x, v) \in \mathbb{R} \times (1/\alpha, 1)$, *we define the function* Ψ *and* $\widetilde{\Psi}$ *by*

$$\Psi(x, v) := \int_{\mathbb{R}} (x-y)_+^{v-1/\alpha} \psi(y) dy, \; \widetilde{\Psi}(x, v) := \frac{d^2}{dx^2} \int_{\mathbb{R}} (y-x)_+^{1/\alpha - v} \psi(y) dy. \tag{5}$$

Proposition 2.1

- Ψ *and* $\widetilde{\Psi}$ *are* $\mathcal{C}^3(\mathbb{R} \times (1/\alpha, 1))$, *and are infinitely differentiable with respect to* v.
- $\Psi, \widetilde{\Psi}$ *as well as all their partial derivatives of any order are well-localized* **in** x **uniformly in** v. *i.e for all* $p \in \{0, 1, 2, 3\}$ *and all* $q \in \mathbb{N}$ *then*

$$\sup_{v \in [1/\alpha, 1]} \sup_{x \in \mathbb{R}} \left(\left| (\partial_x^p \partial_v^q \Psi)(x, v) \right| + \left| (\partial_x^p \partial_v^q \widetilde{\Psi})(x, v) \right| \right) (1 + |x|)^2 < \infty.$$

We denote by $\{\epsilon_{j,k} : (j,k) \in \mathbb{Z} \times \mathbb{Z}\}$ the sequence of Strictly $\alpha-$Stable random variable defined as

$$\epsilon_{j,k} := \int_{\mathbb{R}} 2^{j/\alpha} \psi(2^j s - k) Z_\alpha(ds), \tag{6}$$

Observe that for every fixed integers j, m and r satisfying $m \geq \operatorname{diam}\big(\operatorname{supp}\psi\big)$ and $0 \leq r < m$, $\{\epsilon_{j,r+ml} : l \in \mathbb{Z}\}$ is a sequence of independent random variables; this is a consequence of the fact that the functions $\psi(2^j \cdot -ml)$, $l \in \mathbb{Z}$, have disjoint supports.

Lemma 2.1 (Ayache, Roueff, Xiao'09)

There exists an event Ω_0^* of probability 1, such that for any $\eta > 0$, any $\omega \in \Omega_0^*$ and for all $(j,k) \in \mathbb{Z} \times \mathbb{Z}$, we have

$$|\epsilon_{j,k}(\omega)| \leq C(\omega)(1+|j|)^{1/\alpha+\eta}(1+|k|)^{1/\alpha} \log^{1/\alpha+\eta}(2+|k|), \tag{7}$$

where $C > 0$ is an almost surely finite random variable, only depending on η.

There is no restriction to assume that $\mathcal{K} = [-M, M]$, where M is some positive fixed real-number.

Theorem 2.1

(i) *The field $\{X(u,v) : (u,v) \in \mathcal{K} \times [a,b]\}$ can almost surely be expressed as*

$$X(u,v) = \lim_{n\to+\infty} \sum_{(j,k)\in D_{n,M}} \epsilon_{j,k} 2^{-jv} \Big(\Psi(2^j u - k, v) - \Psi(-k, v) \Big), \quad (8)$$

where $D_{n,M} := \{(j,k) \in \mathbb{Z}^2 : |j| \leq n \text{ and } |k| \leq M2^{n+1}\}$ and where the convergence holds for every $\gamma < a - 1/\alpha$ in the sense of the norm of the Banach space $E_\gamma := \mathcal{C}^1([a,b], \mathcal{C}^\gamma(\mathcal{K}, \mathbb{R}))$ of the Lipschitz functions defined on $[a,b]$ and with values in the Hölder space $\mathcal{C}^\gamma(\mathcal{K}, \mathbb{R})$.

(ii) With probability 1, for all fixed $u \in \mathbb{R}$, $v \mapsto X(u,v)$ is a C^∞ function over $(1/\alpha, 1)$, moreover for each $q \in \mathbb{N}$, the field $\{(\partial_v^q X)(u,v) : (u,v) \in \mathcal{K} \times [a,b]\}$ can almost surely be expressed as

$$(\partial_v^q X)(u,v) = \lim_{n \to +\infty} \sum_{(j,k) \in D_{n,M}} \epsilon_{j,k} \sum_{p=0}^{q} \binom{q}{p} (-j \log 2)^p \, 2^{-jv} \Big((\partial_v^{q-p} \Psi)(2^j u - k, v) - (\partial_v^{q-p} \Psi)(-k, v) \Big),$$

where the convergence holds for every $\gamma < a - 1/\alpha$ in the sense of the norm of the Banach space E_γ.

Any function F in the Banach space E_γ can be viewed as a real-valued function defined on $\mathcal{K} \times [a,b]$ and then $\|F\|$ its norm in this space is equivalent to the norm

$$\sup_{(u,v)\in\mathcal{K}\times[a,b]} |F(u,v)| + \sup_{(u_1,u_2,v)\in\mathcal{K}^2\times[a,b]} \frac{|F(u_1,v)-F(u_2,v)|}{|u_1-u_2|^\gamma}$$
$$+ \sup_{(u,v_1,v_2)\in\mathcal{K}\times[a,b]^2} \frac{|F(u,v_1)-F(u,v_2)|}{|v_1-v_2|}$$
$$+ \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{K}^2\times[a,b]^2} \frac{|F(u_1,v_1)-F(u_1,v_2)-F(u_2,v_1)+F(u_2,v_2)|}{|u_1-u_2|^\gamma|v_1-v_2|}.$$

Corollary 1

There is a random variable $C > 0$ such that a.s. for all $v_1, v_2 \in [a, b]$ one has

$$\sup_{u\in\mathcal{K}} |X(u, v_1) - X(u, v_2)| \leq C\, |v_1 - v_2| \, . \qquad (9)$$

Corollary 2

By replacing (u, v) by $(t, H(t))$ and $[a, b]$ by $[H_*, H^*]$, one can get, in view of (1), a random wavelet series representation of Y, the LMSM, which is a.s. convergent in all the Hölder spaces $C^\gamma(\mathcal{K}, \mathbb{R})$ of order $\gamma < \min\{H_* - 1/\alpha, \beta_H(\mathcal{K})\}$.
Thus, $\beta_Y(\mathcal{K})$, the critical uniform Hölder exponent of the trajectories of Y satisfies a.s. $\beta_Y(\mathcal{K}) \geq \min\{H_* - 1/\alpha, \beta_H(\mathcal{K})\}$.

Sketch of proof

For every fixed $(u,v) \in \mathcal{K} \times [a,b]$, we set $s \mapsto (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}$ belongs to $L^\alpha(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$(u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} = \sum_{j,k\in\mathbb{Z}^2} \kappa_{j,k}(u,v)\psi_{j,k}(s). \tag{10}$$

Using the $L^2(\mathbb{R})-$orthonormality of the sequence $\{2^{j(1/2-1/\alpha)}\psi_{j,k}\,;\,(j,k)\in\mathbb{Z}^2\}$, we have

$$\begin{aligned}\kappa_{j,k}(u,v) &= 2^{j(1-1/\alpha)}\int_{\mathbb{R}}\left((u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}\right)\psi(2^j s-k)ds \\ &= 2^{-jv}\{\Psi(2^j u-k,v)-\Psi(-k,v)\}.\end{aligned}$$

We obtain the following random wavelet serie :

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\epsilon_{j,k}\{\Psi(2^j u-k,v)-\Psi(-k,v)\}. \tag{11}$$

Outline

Proposition 3.1

Let Ω_0^* be the event of probability 1, defined at the Lemma 2.1. Then for every compact set $\mathcal{K} \subset \mathbb{R}$, for all $\omega \in \Omega_0^*$, $q \in \mathbb{N}$, and any arbitrarily small $\eta > 0$, one has :

$$\sup_{(t,s,v)\in\mathcal{K}^2\times[a,b]} \frac{|(\partial_v^q X)(t,v,\omega) - (\partial_v^q X)(s,v,\omega)|}{|t-s|^{v-1/\alpha}\left(1+|\log|t-s||\right)^{q+2/\alpha+\eta}} < \infty. \qquad (12)$$

Theorem 3.1

Let Ω_0^* be the event of probability 1 that will be introduced in the Lemma 2.1. For any arbitrarily small $\eta > 0$ and all $\omega \in \Omega_0^*$, there is a random variable $C > 0$ such that for all $t, s \in \mathcal{K}$, one has

$$|Y(t,\omega) - Y(s,\omega)| \leq C(\omega) \Big\{ |t-s|^{\max\{H(s),H(t)\}-1/\alpha} (1+|\log|t-s||)^{2/\alpha+\eta} + |H(t)-H(s)| \Big\}.$$

$(\mathcal{A})$: there exists $\beta > H^*$ such that $H(\cdot) \in \mathcal{C}^\beta(\mathcal{K}, \mathbb{R})$

Corollary 3.1

Under the condition $(\mathcal{A})$ *one has for all arbitrarly small* $\eta > 0$ *and* $\omega \in \Omega_0^*$,

$$\sup_{t,s \in \mathcal{K}} \frac{|Y(t,\omega) - Y(s,\omega)|}{|t-s|^{\max\{H(s),H(t)\}-1/\alpha} (1 + |\log|t-s||)^{2/\alpha+\eta}} < +\infty,$$

and as consequence

$$\sup_{t,s \in \mathcal{K}} \frac{|Y(t,\omega) - Y(s,\omega)|}{|t-s|^{H_*-1/\alpha} (1 + |\log|t-s||)^{2/\alpha+\eta}} < +\infty.$$

Optimality of modulus of continuity

Theorem 3.2

Let us set $\rho :=$
$\sup\left\{\theta \in \mathbb{R}_+ \ : \ \exists\, t_0 \in \mathcal{K} \text{ satisfying } H(t_0) = H_* \text{ and } \sup_{t\in\mathcal{K}} \frac{|H(t)-H(t_0)|}{|t-t_0|^\theta} < \infty\right\}.$
and $\tau = \frac{1+2\alpha^{-1}}{\alpha\rho-1}$. *Then, under the condition* $(\mathcal{A})$, *for all* $\epsilon > 0$, *one has, almost surely,*

$$\sup_{t,s\in\mathcal{K}} \frac{|Y(t)-Y(s)|}{|t-s|^{H_*-1/\alpha}\left(1+|\log|t-s||\right)^{-\tau-\epsilon}} = \infty. \tag{13}$$

Sketch of proof

$$|X(t,v,\omega) - X(s,v,\omega)| \leq C(\omega) \sum_{j,k\in\mathbb{Z}^2} 2^{-jv}(1+|j|)^{1/\alpha+\eta}(1+|k|)^{1/\alpha}$$
$$\times \log^{1/\alpha+\eta}(2+|k|)|\Psi(2^j t - k, v) - \Psi(2^j s - k, v)|$$

We need to control the following quantity :

$$|\Psi(2^j t - k, v) - \Psi(2^j s - k, v)|$$

1. For all $(j,k) \in \mathbb{Z}^2$, we have

   $$|\Psi(2^j t - k, v) - \Psi(2^j s - k, v)| \leq c_1 \left\{(2+|2^j t - k|)^{-2} + (2+|2^j s - k|)^{-2}\right\}.$$

2. If the following hypothesis holds $2^j|t-s| \leq 1$ then

   $$|\Psi(2^j t - k, v) - \Psi(2^j s - k, v)| \leq c_2 2^j |t-s|(2+|2^j t - k|)^{-2}$$

Introduce j_0 the unique integer such that $1/2 < 2^{j_0}|t-s| \leq 1$

$$|X(t,v,\omega)-X(s,v,\omega)| \leq C_1(\omega)\left(A_{j_0}(t,v)|t-s| + B_{j_0}(t,s,v)\right) \quad (14)$$

with

$$A_{j_0}(t,v) = \sum_{j \leq j_0} 2^{j(1-v)}(1+|j|)^{1/\alpha+\eta} \sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)}{(2+|2^j t-k|)^2}$$

$$\begin{aligned} B_{j_0}(t,s,v) = & \sum_{j \geq j_0+1} \sum_{k\in\mathbb{Z}} 2^{-jv}(1+|j|)^{1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|) \\ & \times \left\{(2+|2^j t-k|)^{-2} + (2+|2^j s-k|)^{-2}\right\} \end{aligned}$$

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