Multifractal formalisms for oscillating singularities

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Joint work with Patrice Abry and Stéphane Roux (ENS de Lyon)

Wavelets and Fractals
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What are oscillating singularities ?

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Signals whose instantaneous frequency changes

Examples :

- Ultrasonic signals emmited by bats
- Gravitational waves
- Structures predicted in fully developed turbulence
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Local singular behaviors of signals
(B. Torresani, Y. Meyer)

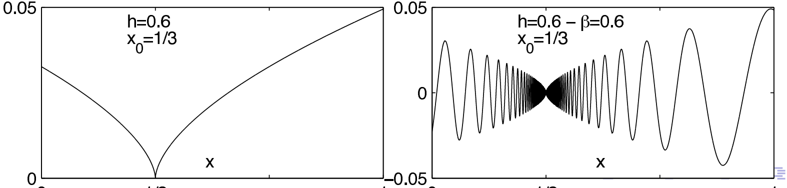
$$f(x) - f(x_0) \sim |x - x_0|^\alpha \sin\left(\frac{1}{|x - x_0|^\beta}\right)$$

Types of pointwise singularities

Cusps : $f(x) - f(x_0) = |x - x_0|^\alpha$

Chirps : $f(x) - f(x_0) = |x - x_0|^\alpha \sin\left(\frac{1}{|x - x_0|^\beta}\right)$

We look for this typical behavior (not the exact form)



Spectrum of singularities

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}.$$

The Iso-Hölder sets of f are

$$E_H = \{x_0 : \quad h_f(x_0) = H\}.$$

The spectrum of singularities of a function f is

$$d_f(H) = dim\,(E_H)$$

where *dim* stands for the Hausdorff dimension.

Wavelet techniques

(initiated by A. Arneodo et al.)

A wavelet basis on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the

$$2^{j/2}\psi(2^j x - k), \qquad j, k \in \mathbb{Z}$$

form an orthonormal basis of $L^2(\mathbb{R})$.

Notations :

Dyadic intervals : $\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$

Wavelet coefficients : $c_{j,k} = c_\lambda = 2^j \int f(x)\psi(2^j x - k)dx$

If $\psi_\lambda(x) = \psi(2^j x - k)$, then $f(x) = \sum_\lambda c_\lambda \psi_\lambda(x)$.

Wavelet leaders

Let λ be a dyadic cube ; 3λ is the cube of same center and three times wider.

Let f be a bounded function ; the wavelet leaders of f are the quantities

$$L_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$

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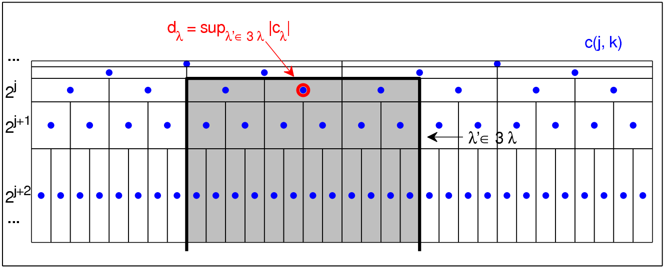
Notations : Let $x_0 \in \mathbb{R}^d$

$\lambda_j(x_0)$ is the dyadic cube of width 2^{-j} which contains x_0

$$L_j(x_0) = L_{\lambda_j(x_0)} = \sup_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|.$$

Wavelet leaders



A function f is uniform Hölder if $f \in C^\varepsilon$ for an $\varepsilon > 0$, i.e.

$$\exists C > 0 : \quad \forall j, \qquad |c_\lambda| \leq C \cdot 2^{-\varepsilon j}.$$

Characterization of pointwise smoothness : If f is uniform Hölder, then

$$\forall x_0 \in \mathbb{R}^d \qquad h_f(x_0) = \liminf_{j \to +\infty} \frac{\log(L_j(x_0))}{\log(2^{-j})}.$$

$$L_j(x_0) \sim 2^{-h_f(x_0) j}$$

Multifractal formalism

(G. Parisi, U. Frisch, A. Arneodo, S. Jaffard, ...)

$$\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$$

$$T_{p,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_\lambda)^p \sim 2^{-\eta_f(p) j}$$

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$\eta_f(p)$ is the leader scaling function

The Legendre spectrum of f is

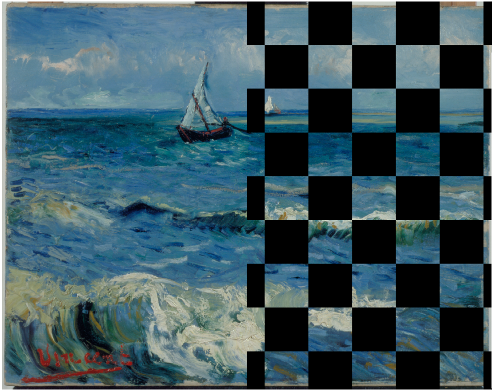
$$\mathcal{L}_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

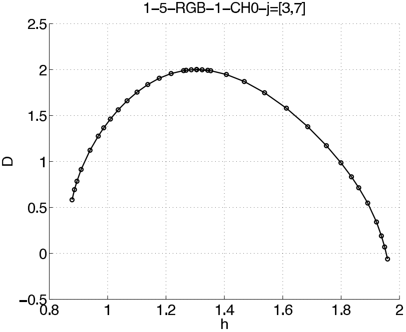
The wavelet leaders multifractal formalism holds if

$$d_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

Multifractal analysis of paintings : The Van Gogh challenge

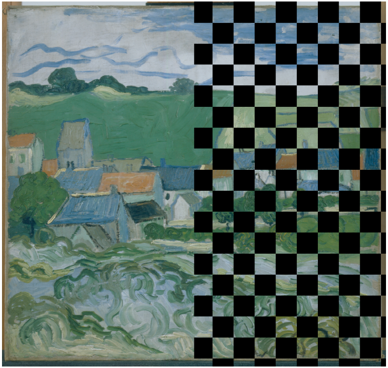
Collaboration with D. Rockmore (Dartmouth) and H. Wendt (Purdue)

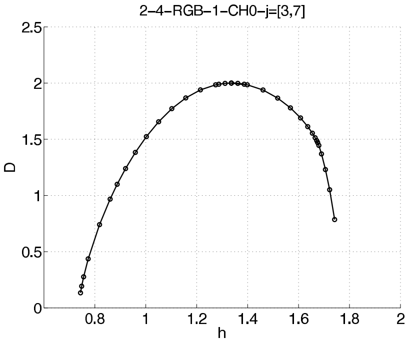


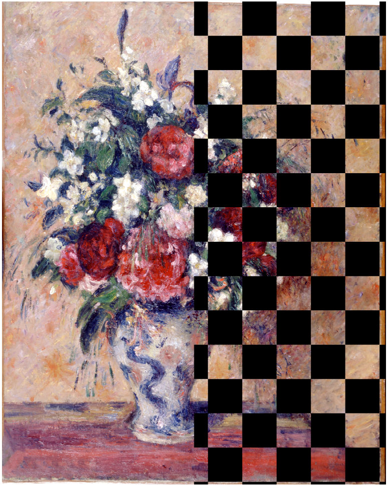


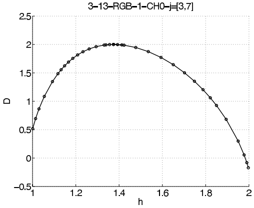
Van Gogh (f415) Arles -Saint Rémy

- f799 (Van Gogh)









Unknown (f248a)

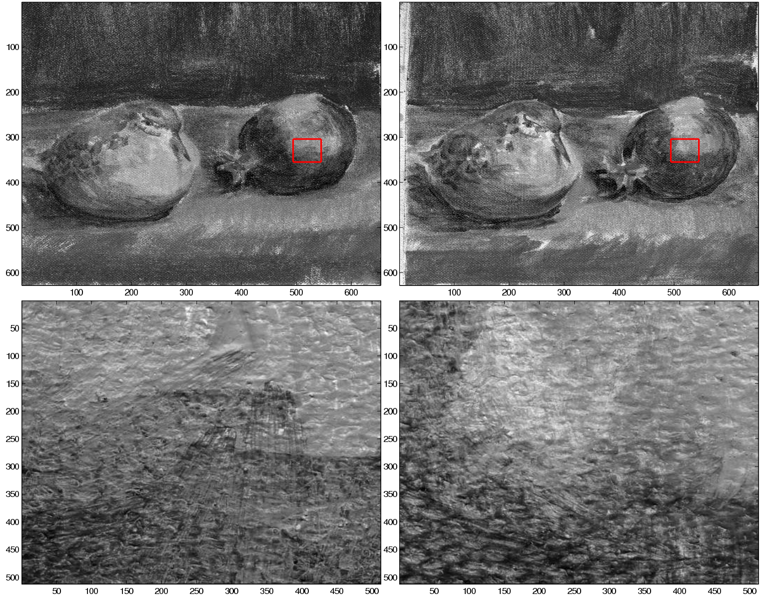
Princeton Experiment

- Experiment design :
  - same Painter (Charlotte Casper) does Original and Copies
  - a series of 7 small paintings,
  - different set of materials (various brushes, grounds, paints)
  - Original and Copies with same set of materials
  - high resolution digitalisation, under uniform conditions.
- Description :

Pair	Ground	Paint	Brushes
1	CP Canvas	Oils	S& H
2	CP Canvas	Acrylics	S& H
3	Smooth CP Board	Oils	S& H
4	Bare linen canvas	Oils	S
5	Chalk and Glue	Oils	S
6	CP Canvas	Acrylics	S
7	Smooth CP Board	Oils	S

Charlotte2's Original & Copy

Charlotte2 MF



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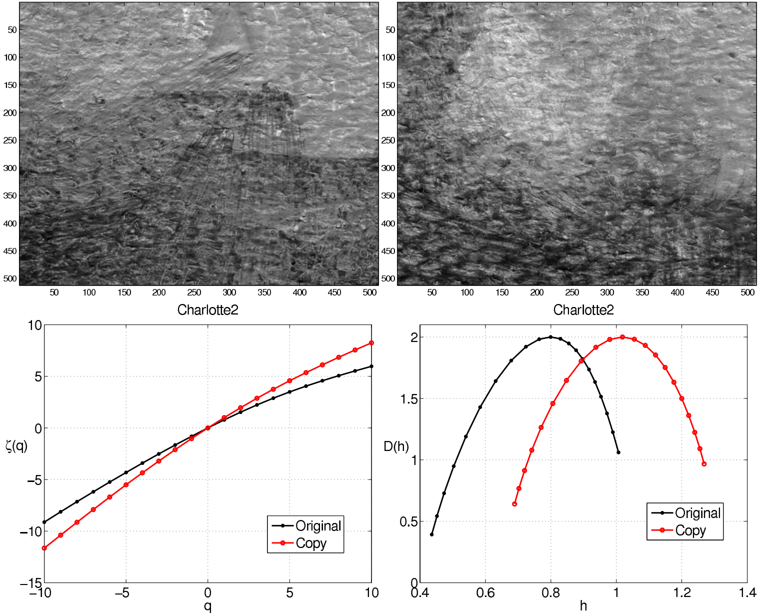

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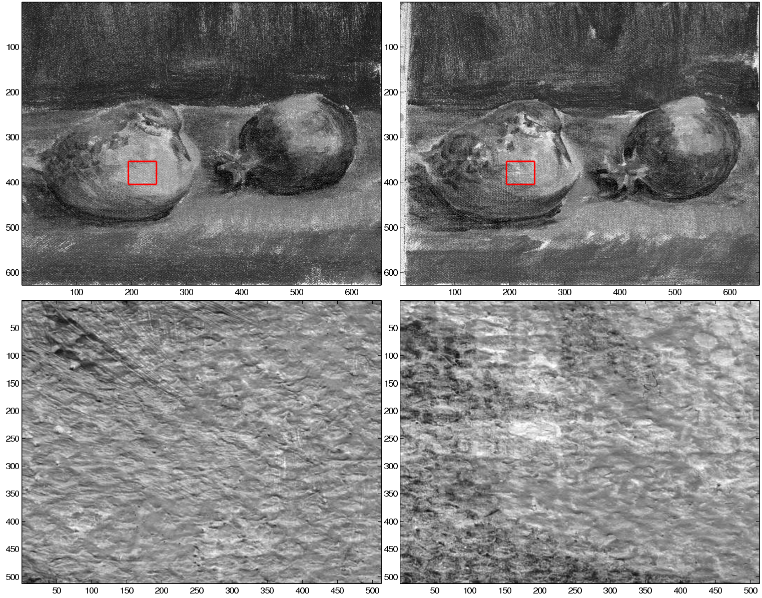

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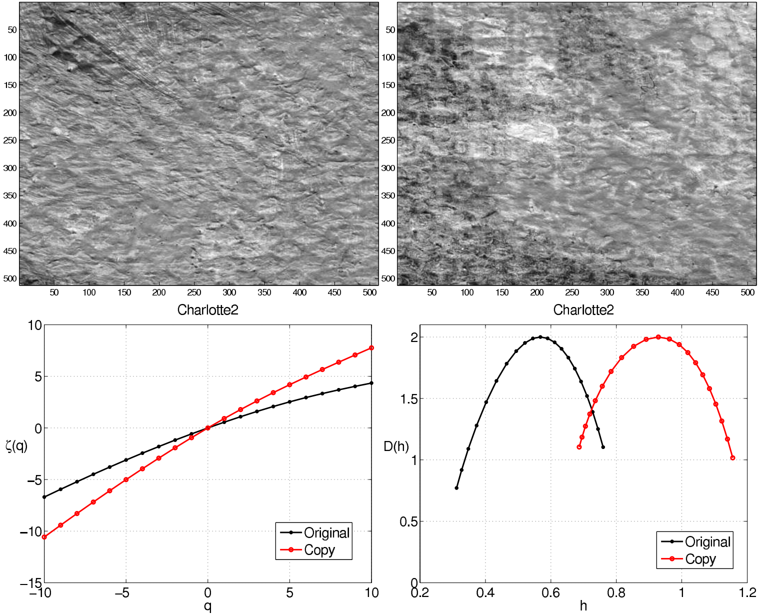

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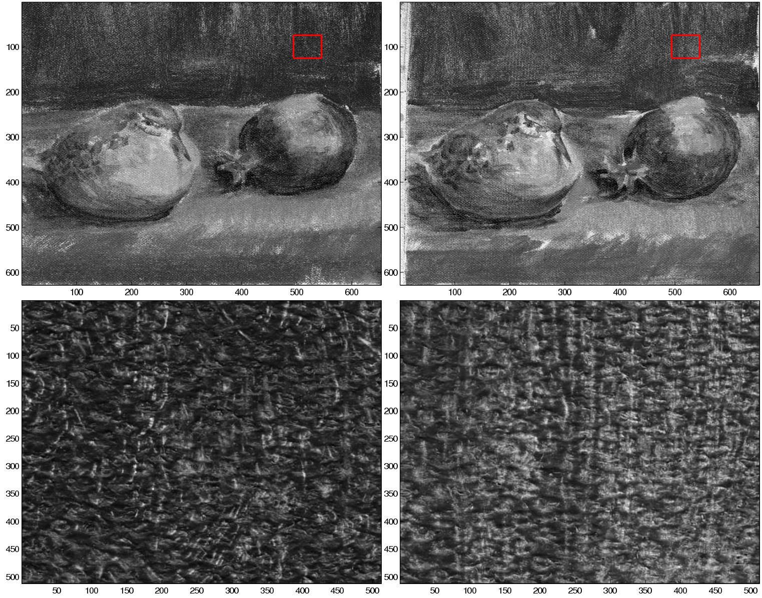

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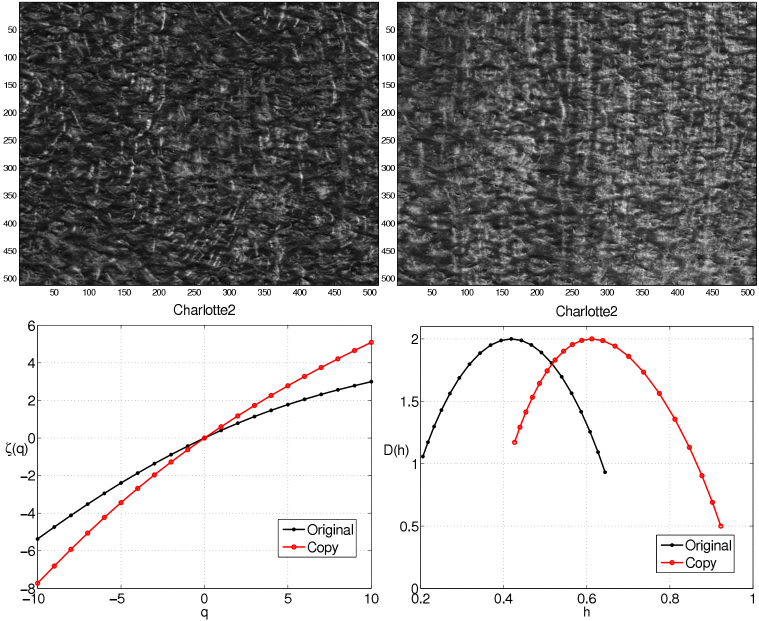


Multifractal analysis of oscillating singularities

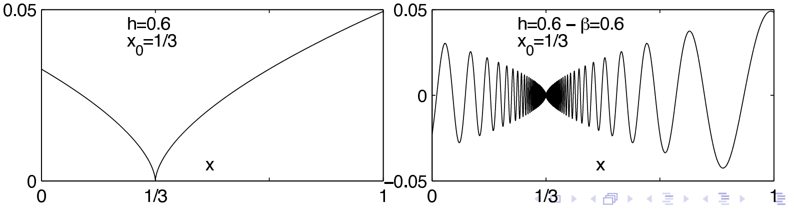
Cusps : $f(x) - f(x_0) = |x - x_0|^H$

After one integration : $f^{(-1)}(x) - f^{(-1)}(x_0) \sim \frac{1}{H}|x - x_0|^{H+1}$

Chirps : $f(x) - f(x_0) = |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)$

After one integration :

$$f^{(-1)}(x) - f^{(-1)}(x_0) = \frac{|x - x_0|^{H+(1+\beta)}}{\beta} \cos\left(\frac{1}{|x - x_0|^\beta}\right) + \cdots$$



Fractional Integration

The fractional integral of order s is the operator $\mathcal{I}^s$ satisfying

$$\widehat{\mathcal{I}^s f}(\xi) = (1 + |\xi|^2)^{-s/2}\, \hat{f}(\xi)$$

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- If f is a cusp at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + s$
- If f is a chirp at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + (1 + \beta)s$

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Definition : Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function.
The oscillation exponent of f at x_0 is

$$\beta_f(x_0) = \left(\frac{\partial (h_{\mathcal{I}^s f}(x_0))}{\partial s} \right)_{s=0^+} - 1$$

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f has a cusp at x_0 if $\beta_f(x_0) = 0$. It follows that

$$h_{\mathcal{I}^s f}(x_0)) = h_f(x_0) + s$$

Pseudo-fractional integration

Heuristic :

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The pseudo-fractional integral of order s, denoted by $\mathcal{J}^s(f)$ consists in replacing the wavelet coefficients c_λ by $2^{-sj} c_\lambda$.

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Theorem : If f is uniform Hölder, then $\mathcal{I}^s(f)$ and $\mathcal{J}^s(f)$ share the same pointwise exponents and spectra

Algorithm :

- Operate pseudo-fractional integration of order s
- Perform the multifractal analysis of this new function

One obtains the integrated Legendre spectrum : $\mathcal{L}^s_f(H) = \mathcal{L}^s_{\mathcal{I}^s(f)}(H)$

Fractionally integrated spectra

Definition : The integrated spectrum of f is

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If all points are cusps, then the integrated spectrums of singularities are shifted

$$d_f^s(H) = d_f(H - s)$$

Heuristics :

If all points are cusps, then the integrated Legendre spectrums are shifted

Fractionally integrated spectra

Definition : A uniformly Hölder function f is of cusp type

$$\forall \lambda, \quad \exists \lambda' \subset \lambda \quad j' = j + o(j) \;\; \text{et} \;\; |c_{\lambda'}| \geq L_\lambda 2^{-o(j)}$$

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Proposition : Si f is uniformly of cusp type then the integrated Legendre spectra satisfy

$$\mathcal{L}_f^s(H) := \mathcal{L}_{\mathcal{I}^s f}(h) = \mathcal{L}_f(H - s)$$

Exemples : FBM, Random Wavelet cascades of A. Arneodo, The measure-based random wavelet series of J. Barral and S. Seuret,...

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Heuristics :

Cusps= Signature of clustering of large wavelet coefficients
Oscillating singularities= Dispersion of large wavelet coefficients

Problem : Oscillating singularities can be present even if the integrated Legendre spectra are shifted

The β-spectrum

The iso-oscillating sets are

$$F_\beta = \{x : \beta_f(x) = \beta\}$$

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Heuristic : Let s "small enough " be given. If f has an oscillating singularity of exponents (H, β) at x_0, then

$$L_{\lambda_j(x_0)}(f)(x_0) \sim 2^{-Hj} \quad \text{and} \quad L_{\lambda_j(x_0)}(\mathcal{I}^s f) \sim 2^{-(H+s(1+\beta))j}$$

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The β-leaders by

$$B_\lambda = 2^j \left(\frac{L_{\lambda_j(x_0)}(\mathcal{I}^s f(x_0))}{L_{\lambda_j(x_0)}(f)(x_0)} \right)^{1/s} \sim 2^{-\beta j}$$

The β-formalism

The β-structure function is $B_{p,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} (B_\lambda)^p$

The β-scaling function is

$$\omega_f(p) = \liminf_{j \to +\infty} \frac{\log(B_{p,j})}{\log(2^{-j})}$$

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$$\mathcal{D}_f(\beta) = \inf_{p \in \mathbb{R}} \left(d + \beta p - \omega_f(p)\right)$$

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Theorem :

- If f is of cusp-type, then $\mathcal{D}_f(\beta)$ is supported by a point
- The β-formalism holds for lacunary wavelet series

The grandcanonical formalism

Spectrum of oscillating singularities

$$\mathbb{D}_f(H,\beta) = dim(\{x_0 : \quad h_f(x_0) = H \quad \text{et} \quad \beta_f(x_0) = \beta\})$$

The grandcanonical formalism

Spectrum of oscillating singularities

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If f has an oscillating singularity of exponants (H, β) at x_0, then

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The grandcanonical structure function is

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The grandcanonical multifractal formalism holds if

$$\mathbb{D}_f(H,\beta) = \inf_{p,q \in \mathbb{R}\times\mathbb{R}} (d + Hp + \beta q - \Omega_f(p,q))$$

Theorem :

The grandcanonical multifractal formalism holds for cusp-type functions and for lacunary wavelet series

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Challenges :

Turbulence

Oscillating singularities on sets of small dimension

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Challenges :

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Alternative stochastic processes which display oscillating singularities

Heuristic : oscillating singularities are present when wavelet coefficients are both sparse and dispersed without interscale correlations