Some studies on dynamical diophantine approximation

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This talk concerns the following two works :

1. **The dimension set of recurrence of the irrational rotation**, (with Dong Han Kim), work in progress.
2. **Dynamical Diophatine approximation for Markov interval maps**, (with Stéphane Seuret), preprint.

Outline

Introduction

I. General context

(X, d) complete metric space.

- $\{x_n\}_{n\geq 1}$ a sequence of points in X.
- $\{r_n\}$ a sequence of positive real numbers.

Define

- $\mathcal{L}(\{x_n\}, \{r_n\}) := \limsup B(x_n, r_n) = \{y : \ d(y, x_n) < r_n \ \ i.o.\}$
  (i.o. = infinitely often)
- $\mathcal{F}(\{x_n\}, \{r_n\}) := X \setminus \limsup B(x_n, r_n) = \{y : \ d(y, x_n) \geq r_n \ \ ev.\}$
  (ev. = eventually)

Question :

What are the sizes of $\mathcal{L}(\{x_n\}, \{r_n\})$ and $\mathcal{F}(\{x_n\}, \{r_n\})$?

$\rightarrow$ Their measures (If there is a measure μ on X) ?

$\rightarrow$ Their Hausdorff dimensions ?

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II. Three examples

- $X = \mathbb{R}$, $\{x_n\} = \mathbb{Q}$:

$$\left\{ y : |y - \frac{p}{q}| < \frac{1}{q^\kappa} \ i.o. \right\}$$

- X is a probability space and $\{x_n\}$ is a random sequence.
  Example : $\{x_n\}$ is an i.i.d. sequence

$$\left\{ y : |y - x_n| < \frac{1}{n^\kappa} \ i.o. \right\}$$

- (X, T) is a dynamical system and $x_n = T^n x$.
  Example : $X = \mathbb{R}/\mathbb{Z}$, $Tx = 2x \bmod 1$.

$$\left\{ y : |y - T^n x| < \frac{1}{n^\kappa} \ i.o. \right\}.$$

III. Independent and uniformly distributed sequence

- $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $\omega = \{\omega_n\}_{n\geq 1}$ is an independent and uniformly distributed sequence of $\mathbb{T}$.
- r_n is a decreasing sequence of positive real numbers tending to 0.

We are interested in the set :

$$\mathcal{L}(\omega) := \{y \in \mathbb{T} : |\omega_n - y| < \frac{r_n}{2} \quad i.o.\} = \limsup_{n\to\infty} \left]\omega_n - \frac{r_n}{2}, \omega_n + \frac{r_n}{2}\right[$$

$\rightarrow$ points covered by infinite random intervals $]-\frac{r_n}{2}, \frac{r_n}{2}[+\omega_n$.

Question of **Dvoretzky 1956** : When almost surely $\mathcal{L}(\omega) = \mathbb{T}$?

Case $r_n = c/n$:

Kahane 1959 : $\mathcal{L}(\omega) = \mathbb{T}$ $a.s.$ if $c > 1$.

Erdös conjecture : $\mathcal{L}(\omega) = \mathbb{T}$ $a.s.$ if and only if $c \geq 1$.

Billard 1965 : For $c < 1$, No.

Mandelbrot 1972, Orey (independently) : $\mathcal{L}(\omega) = \mathbb{T}$ $a.s.$ if $c = 1$.

IV. Independent and uniformly distributed sequence

General case :

Shepp 1972 : $\mathcal{L}(\omega) = \mathbb{T}$ $a.s.$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{r_1 + \cdots + r_n} = \infty.$$

Dimension results :

Kahane : If $r_n = c/n (c < 1)$, then almost surely

$$\dim_H (X \setminus \mathcal{L}(\omega)) = 1 - c$$

Fan-Wu 2004 : If $r_n = c/n^{\kappa}$,$c > 0$ and $\kappa > 1$, then almost surely

$$\dim_H \mathcal{L}(\omega) = 1/\kappa$$

Further and related studies :

Barral, Fan, Kahane, Jaffard ...

V. Sequence of processes

An independent sequence not uniformly distributed ?

Jonasson-Steif 2008 :
Brownian motion and Poisson process.

- $x_n = U_{n,t}(\omega)$ depends not only on ω and but also on the time t.
- $r_n = c/n$ or $r_n = c/n^\kappa$.

Dynamical Diophantine approximation

I. Dynamical Diophantine approximation

- (X, d) is a complete metric space and $T : X \to X$.
- $x_n = T^n x$, $r_n = 1/n^\kappa$.

We are interested in three types of sets :

$$\mathcal{L}_1^\kappa(x) := \{y \in X : d(T^n x, y) < 1/n^\kappa \quad i.o.\},$$

$$\mathcal{L}_2^\kappa(y) := \{x \in X : d(T^n x, y) < 1/n^\kappa \quad i.o.\},$$

or if we have a family of transformations $\{T_\theta\}_{\theta\in\Theta}$

$$\mathcal{L}_3^\kappa(x, y) := \{\theta \in \Theta : d(T_\theta^n x, y) < 1/n^\kappa \quad i.o.\}.$$

Remark : The sets $B(T^n x, 1/n^\kappa)$ are "moving targets" and $y \in B(T^n x, 1/n^\kappa) \Leftrightarrow y$ "hitting one target".

II. Relation with the hitting time

For $x, y \in X$, define the hitting time of x in the ball $B(y, r)$ by

$$\tau_r(x, y) := \inf\{n \geq 1 : T^n x \in B(y, r)\}.$$

Set

$$\underline{R}(x, y) := \liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r}.$$

Denote the orbit of x

$$\mathcal{O}(x) = \{T^n x : n \geq 0\}, \quad \mathcal{O}^+(x) = \mathcal{O}(x) \setminus \{x\}.$$

Lemma

For $\kappa > 0$, we have :

$$\left\{y \in X : \underline{R}(x, y) < \frac{1}{\kappa}\right\} \setminus \mathcal{O}^+(x) \subset \mathcal{L}_1^\kappa(x) \subset \left\{y \in X : \underline{R}(x, y) \leq \frac{1}{\kappa}\right\}.$$

Irrational rotation

I. Dirichlet, Khintchine, Duffin-Schaefer

Suppose $X = \mathbb{T}$. For $\theta \in \mathbb{R}$, define the rotation of θ on X by :

$$T_\theta : x \mapsto x + \theta \pmod 1.$$

Dirichlet :

$$\left\{\theta \in \mathbb{R} : \|n\theta\| < \frac{1}{n}\ i.o.\right\} = \mathbb{R}\ (\Leftrightarrow \mathcal{L}_3^1(0,0) = \mathbb{R}).$$

Khintchine : Suppose $\Psi : \mathbb{N} \to \mathbb{R}^+$ is a decreasing function. Consider :

$$\mathcal{K}(\Psi) := \{\theta : \|n\theta\| < \Psi(n)\ i.o.\}\ (= \mathcal{L}_3^\Psi(0,0))$$

- $\mathcal{K}(\Psi)$ is of Lebesgue measure zero if $\sum \Psi(n) < \infty$;
- $\mathcal{K}(\Psi)$ is of full Lebesgue measure if $\sum \Psi(n) = \infty$.

Conjecture of **Duffin-Schaefer** (1941) : If Ψ is not decreasing, then $\mathcal{K}(\Psi)$ is of full Lebesgue measure if $\sum \phi(n)\Psi(n)/n = \infty$ where ϕ is the Euler function.

Haynes-Pollington-Velani (2009) : True if $\sum \phi(n)(\Psi(n)/n)^{1+\epsilon} = \infty$.

II. Hausdorff Dimension

Jarník 1929, Besicovith 1934 :

$$\dim_H \mathcal{L}_3^\kappa(0,0) = \dim_H \left\{\theta : \|n\theta\| < \frac{1}{n^\kappa} \ i.o.\right\} = \begin{cases} 2/(\kappa+1) & \kappa > 1 \\ 1 & \kappa \leq 1 \end{cases}.$$

Levesley 1998 : For all $y \in \mathbb{R}$, $\kappa > 1$:

$$\dim_H \mathcal{L}_3^\kappa(0,y) = \dim_H\{\theta : ||n\theta - y|| < n^{-\kappa} \ \ i.o.\} = 2/(\kappa+1).$$

Bugeaud 2003, Troubetzkoy-Schmeling 2003 : for all $\theta \in \mathbb{R}$, $\kappa > 1$:

$$\dim_H \mathcal{L}_1^\kappa(0) = \dim_H\{y : ||n\theta - y|| < n^{-\kappa} \ \ i.o.\} = 1/\kappa.$$

III. Hausdorff dimension (variations)

Troubetzkoy-Schmeling 2003 : Consider

$$\mathcal{B}_y(\ell, m) := \{\theta : ||n\theta - y|| < n^{-\kappa} \quad i.o. \quad n \equiv \ell \pmod{m}\}$$

and

$$\mathcal{A}_\theta(\ell, m) := \{y : ||n\theta - y|| < n^{-\kappa} \quad i.o. \quad n \equiv \ell \pmod{m}\}.$$

Then

$$\dim_H(\cap_{\ell=0}^{m-1}\mathcal{B}_y(\ell, m)) = \frac{2}{\kappa+1} \quad \text{and} \quad \dim_H(\cap_{\ell=0}^{m-1}\mathcal{A}_\theta(\ell, m)) = \frac{1}{\kappa}.$$

Borosh-Fraenkel 1972 :

- $\mathcal{N} \subset \mathbb{N}^+$, $\sharp(\mathcal{N}) = \infty$.
- ν_0 such that $\sum_{n\in\mathcal{N}} n^{-\nu_0} = \infty$ and $\sum_{n\in\mathcal{N}} n^{-\nu_0-\epsilon} < \infty$ for any $\epsilon > 0$

$$\dim_H \left\{\theta : \|n\theta\| < \frac{1}{n^\kappa} \quad i.o. \quad n \in \mathcal{N}\right\} = \min\left\{\frac{1+\nu_0}{\kappa+1}, 1\right\}.$$

IV. Dirichlet again

Dirichlet : Let θ, Q two real numbers with $Q \geq 1$. There exists an integer n such that $1 \leq n \leq Q$, and

$$\|n\theta\| < \frac{1}{Q}.$$

An inhomogeneous analogy of Dirichlet Theorem ?

No ! **Not** for all $y \in \mathbb{T}$,

$$\mathcal{U}_3^1(y) := \left\{\theta : \ \forall Q \geq 1, \ \|n\theta - y\| < Q^{-1} \text{ for some } 1 \leq n \leq Q\right\} = \mathbb{R}.$$

In general, we consider the sets :

$$\mathcal{U}_3^\kappa(y) := \left\{\theta : \ \forall Q \geq 1, \ \|n\theta - y\| < Q^{-\kappa} \text{ for some } 1 \leq n \leq Q\right\},$$

and

$$\mathcal{U}_1^\kappa(0) := \left\{y : \ \forall Q \geq 1, \ \|n\theta - y\| < Q^{-\kappa} \text{ for some } 1 \leq n \leq Q\right\}.$$

VI. Relation with the hitting time ?

Recall : For $x, y \in X$, the hitting time of x in the ball $B(y,r)$ is defined by

$$\tau_r(x,y) := \inf\{n \geq 1 : T^n x \in B(y,r)\}.$$

Set

$$\overline{R}(x,y) := \limsup_{r\to 0} \frac{\log \tau_r(x,y)}{-\log r}.$$

Define

$$\tilde{\mathcal{U}}_1^\kappa(0) := \left\{y : \ \forall Q \gg 1, \ \|n\theta - y\| < Q^{-\kappa} \text{ for some } 1 \leq n \leq Q\right\}.$$

Lemma

For $\kappa > 0$, we have :

$$\left\{y \in X : \overline{R}(0,y) < \frac{1}{\kappa}\right\} \subset \tilde{\mathcal{U}}_1^\kappa(0), \quad \mathcal{U}_1^\kappa(0) \subset \left\{y \in X : \overline{R}(0,y) \leq \frac{1}{\kappa}\right\}.$$

VII. Our small result

An irrational θ, $0 < \theta < 1$, is called of type η if

$$\eta = \sup\{\beta : \liminf_{j\to\infty} j^{\beta}\|j\theta\| = 0\},$$

Theorem (Liao-Kim 2010)

Let θ be an irrational of type η. For $1 \le \beta \le \eta$, we have

$$\dim_H \left\{ y : \limsup_{r\to 0} \frac{\log \tau_r(0,y)}{-\log r} \le \beta \right\} \ge \frac{\beta - 1}{(\eta - 1)(\eta + 1 - \beta)}.$$

VIII. Our small result-Method

Denote $\{q_k\}_{k\geq 1}$ the denominators of the convergents of the continued fraction expansion of θ. Since θ is of type η, for any $\epsilon > 0$, we have

$$\|q_k\theta\| > \frac{1}{q_k^{\eta+\epsilon}},\ \forall k \gg 1 \quad \text{and} \quad \|q_{k_i}\theta\| < \frac{1}{q_{k_i}^{\eta-\epsilon}},\ \exists\{k_i\}.$$

For $k \geq 1$, if $k \neq k_i$, define $F_k = [0,1]$ and if $k = k_i$ define

$$F_{k_i} := \bigcup_{1\leq n\leq \lfloor q_{k_i}^{\beta}\rfloor+1} \big(-(n+q_{k_i})\theta, -(n-q_{k_i})\theta\big).$$

Set $E_0 := [0,1]$ and for $i \geq 1$,

$$E_i := F_1 \cap \cdots \cap F_{k_i} = F_{k_1} \cap F_{k_2} \cap \cdots \cap F_{k_i}.$$

Define $F := \cap E_i$. Then F is a Cantor type subset.

Expanding Markov maps

I. Expanding Markov maps

Let $X = [0,1]$. T is an expanding Markov map :

1. (Expansion property) $\exists$ an integer $n \geq 1$ and a real number ρ such that $|(T^n)'| \geq \rho > 1$,
2. (Piecewise monotonicity) T is strictly monotonic and extends to a C^2 function on each $\overline{I(i)}$,
3. (Markov property) if $I(j) \cap T(I(k)) \neq \emptyset$, then $I(j) \subset T(I(k))$,
4. (Mixing) $\exists$ a R such that $I(j) \subset \cup_{n=1}^{R} T^n(I(k))$ for every k and j,
5. (Rényi's condition)

$$\sup_{I(k)} \sup_{y,z \in I(k)} \frac{|T''(x)|}{|T'(y)||T'(z)|} < \infty.$$

Recall : $\mathcal{L}_1^\kappa(x) := \{y \in X : d(T^n x, y) < 1/n^\kappa \quad i.o.\}$.
Two Hölder functions $\phi, \psi \to$ two Gibbs measures μ_ϕ, ν_ψ.

Questions :

- When $\mu_\phi - a.s.\ x$, $\quad \nu_\psi(\mathcal{L}_1^\kappa(x)) = 1$?
- When $\mu_\phi - a.s.\ x$, $\quad \mathcal{L}_1^\kappa(x) = [0,1]$?
- For $\mu_\phi - a.s.\ x$, $\quad \dim_H(\mathcal{L}_1^\kappa(x)) =?$

II. Results-Notations

Denote :

$$\mathcal{E}_{\mu_\phi}(\alpha) = \{y \in [0,1] : d_\mu(y) = \alpha\}, \ \forall \alpha > 0.$$

multifractal spectrum of μ_ϕ :

$$D_{\mu_\phi} : \alpha \geq 0 \ \longmapsto \ \dim_H \mathcal{E}_{\mu_\phi}(\alpha).$$

Denote $\mathcal{M}$ the set of invariant measures, and

$$h_- := \inf_{\mu \in \mathcal{M}} \frac{\int (-\phi) d\mu}{\int \log |T'| d\mu}, \quad h_{\max} := \frac{\int (-\phi) d\nu}{\int \log |T'| d\nu}, \quad h_+ := \sup_{\mu \in \mathcal{M}} \frac{\int (-\phi) d\mu}{\int \log |T'| d\mu}$$

where ν is the measure associated to $-\log |T'|$.

Denote $\dim_H \mu$ the Hausdorff dimension of μ defined by

$$\dim_H \mu = \inf\{\dim_H E : E \text{ Borel} \subset [0,1] \text{ and } \mu(E) > 0\}.$$

III. Results-Theorem

Theorem (Liao-Seuret 2010)

For μ_ϕ almost all $x \in [0,1]$, we have

1. *if $\kappa > 0$, then*

$$f(1/\kappa) = \dim_H \mathcal{L}_1^\kappa(x) = \begin{cases} \frac{1}{\kappa} & \text{if } \frac{1}{\kappa} \le \dim_H \mu_\phi, \\ D_{\mu_\phi}(\frac{1}{\kappa}) & \text{if } \dim_H \mu_\phi < \frac{1}{\kappa} \le h_{\max}, \\ 1 & \text{if } \frac{1}{\kappa} > h_{\max}. \end{cases}$$

2. *if $h_{\max} \le 1/\kappa \le h_+$, then Leb$(\mathcal{L}_1^\kappa(x)) = 1$.*
3. *if $1/\kappa > h_+$, then $\mathcal{L}_1^\kappa(x) = [0,1]$.*

Fan-Schmeling-Troubetzkoy 2007 : The case $Tx = 2x \pmod 1$.
Differences and difficulties :

- The exponent $\int \log |T'| d\mu$.
- We used a result of **Barral-Seuret 2007** to obtain the lower bound of the first part of the spectrum $f(1/\kappa)$.

Graph of the spectrum

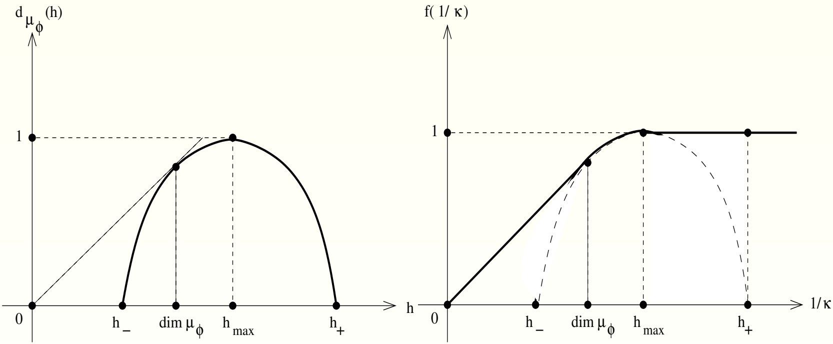

FIG.: spectra of the measure μ_ϕ and $\mathcal{L}_1^\kappa(x)$

IV. Properties of $\underline{R}(x,y)$

It is reduced to determine (for μ_ϕ-almost all x), the spectrum

$$\dim_H \left\{ y \in X : \underline{R}(x,y) < \frac{1}{\kappa} \right\}.$$

Remark that **Feng-Wu 2001, Saussol-Wu 2003** : for all $\alpha \le \beta$

$$\dim_H \{ x \in X : \underline{R}(x,x) = \alpha, \ \text{ and } \ \overline{R}(x,x) = \beta \} = 1.$$

Theorem (Galatolo 2007)

If (X, T, μ) has a super-polynomial decay of correlations and if $\mathrm{d}_\mu(y)$ exists, then for μ-almost all x,

$$\underline{R}(x,y) = \mathrm{d}_\mu(y).$$

Corollary

$$\sup \{ \kappa : \nu_\psi(\mathcal{L}_1^\kappa(x)) = 1 \ \textit{for}\ \mu_\phi\textit{-almost all}\ x \} = \frac{\int_X \log |T'| \, d\nu_\psi}{\int_X (-\phi) \, d\nu_\psi}.$$

V. Others : Badly approximated points

$X = [0, 1]$, $Tx = 2x \pmod 1$, $r_n = c \in [0, 1]$.
Consider the sets :

$$E_c := \{x : ||T^n x|| > c \quad \forall n\} \quad (\approx [0, 1] \setminus \mathcal{L}_2^{r_n = c}(0))$$

Nilsson 2009 : The property of the spectum $f(c) := \dim_H E_c$:

- f is continuous
- $f' = 0 \quad Leb - a.s.$
- $f(c) = 0$ if $c \geq c_0$ where c_0 is determined by the Thue-Morse sequence.
- The graph of f is a **Devil's staircase**.

Thank you for your attention !