Local behavior of traces of Besov functions: prevalent results

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Definition

Given a real function $f \in L^\infty_{\text{loc}}(\mathbb{R}^D)$ and $x_0 \in \mathbb{R}^D$, f is said to belong to $\mathcal{C}^\alpha(x_0)$, for some $\alpha \geq 0$, if there exists a polynomial P of degree at most $\lfloor \alpha \rfloor$ and a constant $C > 0$ such that locally around x_0 :

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

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Spectrum of singularities :

$$d_f : h \in [0, \infty] \longmapsto \mathbf{dim}_{\mathcal{H}} \ E_f(h), \quad \textbf{where } E_f(h) := \{x_0 \in \mathbb{R}^D : h_f(x_0) = h\}.$$

Here $\mathbf{dim}_{\mathcal{H}}$ stands for the Hausdorff dimension.

$0 < d < D$ are two fixed integers. Let $d' := D - d$ and $X = (x, x') \in \mathbb{R}^d \times \mathbb{R}^{d'} = \mathbb{R}^D$. For $a \in \mathbb{R}^{d'}$ we shall denote by $\mathcal{H}_a := \{(x, a)\}$ the d-dimensional affine subspace of $\mathbb{R}^D$. Let f be a continuous function on $\mathbb{R}^D$. Its trace on $\mathcal{H}_a$ is

$$\begin{aligned} f_a := f_{|\mathcal{H}_a} : \ & \mathbb{R}^d \longrightarrow \mathbb{R} \\ & x \longmapsto f(x, a) \end{aligned}$$

Goal : to obtain an upper and a lower bound of the spectrum

The space E is endowed with its Borel σ-algebra $\mathcal{B}(E)$ and all Borel measures μ on $(E, \mathcal{B}(E))$ will be automatically *completed*
A set is said to be *universally measurable* if it is measurable for any (completed) Borel measure.

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Definition

A universally measurable set $A \subset E$ is called *shy* if there exists a Borel measure μ that is positive on some compact subset K of E and such that

$$\text{for every } x \in E, \quad \mu(A + x) = 0.$$

More generally, a set that is included in a shy universally measurable set is also called shy.
Finally, the complement in E of a shy subset is called *prevalent*.

The measure μ used to show that some subset is shy or prevalent is called a *probe*.

Properties

- when a set B is prevalent, it is dense in E
- $B + x$ is also prevalent for any $x \in E$
- if $(B_n)_{n\in\mathbb{N}}$ is a sequence of prevalent sets then so is $\bigcap_{n\in\mathbb{N}} B_n$
- when E has finite dimension, B is prevalent in E if and only if it has full Lebesgue measure.

How to prove that a universally measurable set A is shy?

We set the probe space P to be the d_1-dimensional subspace of E spanned by the functions g^i.

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Take an arbitrary $f \in E$ and for each $\beta \in \mathbb{R}^{d_1}$ define

$$f^\beta := f + \sum_{i=1}^{d_1} \beta_i g^i .$$

Proposition

If for any $f \in E$, the set $\left\{\beta \in \mathbb{R}^{d_1} : f^\beta \in A\right\}$ has d_1-dimensional Lebesgue measure $\mathcal{L}_{d_1}$ equal to 0 then A is shy

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Indeed, if the proposition is true, let us denote by μ the measure $\mathcal{L}_{d_1}$ carried by $\mathcal{P}$ and fix any $f \in E$. For μ-almost $F \in \mathcal{P}$, we know that $f + F \notin A$. Hence the set $\{f + A\} \cap \mathcal{P}$ has a μ-measure equal to 0, i.e.

$$\mu(\{f + A\}) = 0.$$

Since this is true for any $f \in E$, by definition, the set A is shy.

For every $\lambda^D := (j, \mathbf{k}, \mathbf{l}) \in \mathbb{Z} \times \mathbb{Z}^D \times \{0,1\}^D$, we define the tensorized wavelet

$$\Psi_\lambda(x) := \prod_{i=1}^{D} \Psi^{l_i}_{j,k_i}(x_i),$$

with $\mathbf{k} = (k_1, k_2, \cdots, k_D) \in \mathbb{Z}^D$ and $\mathbf{l} = (l_1, l_2, \cdots, l_D) \in \{0,1\}^D$.

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Any function $f \in L^2(\mathbb{R}^D)$ can be written

$$f(X) = \sum_{\lambda^D = (j,\mathbf{k},\mathbf{l}):\, j \in \mathbb{Z},\, \mathbf{k} \in \mathbb{Z}^D,\, \mathbf{l} \in L^D} c_{\lambda^D} \Psi_{\lambda^D}(X),$$

where $L^D := \{0,1\}^D \setminus \{0, .., 0\}$

Theorem (S.Jaffard)

Assume that the wavelet Ψ is regular enough. Let $f : [0,1]^d \to \mathbb{R}$ be a locally bounded function with wavelet coefficients $\{c_\lambda\}$, and let $x \in [0,1]^d$.
If $f \in C^\gamma(x)$, then there exists a constant $M < \infty$ such that for all $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \Lambda^d \times L^d$,

$$|c_\lambda| \leq M\big(2^{-j} + \big|x - k2^{-j}\big|\big)^\gamma = M2^{-j\gamma}(1 + \big|2^j x - k\big|)^\gamma \tag{1}$$

Reciprocally, if (1) holds true and if $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^d)$, then $f \in C^{\gamma-\eta}(x)$, for every $\eta > 0$.

where

$$\begin{aligned} \text{for } j \geq 1, \quad \mathbb{Z}_j &= \{0, 1, \cdots, 2^j - 1\} \quad \text{and} \quad \Lambda_j^d = \{j\} \times \mathbb{Z}_j^d \\ \Lambda^d &= \bigcup_{j \geq 1} \Lambda_j^d. \end{aligned}$$

Definition

Let $0 < s < \infty$, $0 < p, q \leq \infty$. Assume that the wavelet Ψ is regular enough. The $B^s_{p,q}([0,1]^D)$ Besov norm (quasi-norm when $p < 1$ or $q < 1$) of a function f on $[0,1]^D$ having wavelet coefficients c_{λ^D} is defined as

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j \geq 1} \left(2^{(sp-D)j} \sum_{(\mathbf{k},\mathbf{k}') \in \mathbb{Z}^D_j} |c_{\lambda^D}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{2}$$

with the obvious modifications when $p = \infty$ or $q = \infty$.

Theorem (S.Jaffard)

Let $0 < p < \infty$ *and* $D/p < s < \infty$*. For any* $g \in B^s_{p,\infty}(\mathbb{R}^D)$*, for all* $h \geq s - D/p$*,*

$$d_g(h) \leq \min(D, D + (h - s)p),$$

and $E_f(h) = \emptyset$ *if* $h < s - D/p$*.*

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Theorem (A.Fraysse, S.Jaffard)

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < s - D/p < \infty$. For almost all $g \in B^s_{p,q}(\mathbb{R}^D)$,

$$d_g(h) = \begin{cases} D + (h-s)p & \text{if } h \in [s - D/p, s] \\ -\infty & \text{else} \end{cases}$$

and for x in a set of full Lebesgue measure in $\mathbb{R}^D$, $h_g(x) = s$.

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Hence we will consider functions f of the form

$$f(X) = \sum_{\lambda^D \in \Lambda^D \times L^D} c_{\lambda^D} \Psi_{\lambda^D}(X),$$

where

$$\begin{array}{rcl} X & = & (x, x') \\ \text{for } j \geq 1, \quad \mathbb{Z}_j & = & \{0, 1, \cdots, 2^j - 1\} \quad \text{and} \quad \Lambda_j^D = \{j\} \times \mathbb{Z}_j^D \\ \Lambda^D & = & \bigcup_{j \geq 1} \Lambda_j^D. \end{array}$$

$$f_a(x) = \sum_{\lambda^D \in \Lambda^D \times L^D} c_{\lambda^D} \prod_{i=1}^{d} \Psi_{j,k_i}^{l_i}(x_i) \prod_{i=1}^{d'} \Psi_{j,k'_i}^{l'_i}(a_i)$$

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Problem : $(0,1)^D \backslash 0^D \neq (0,1)^d \backslash 0^d \times (0,1)^{d'} \backslash 0^{d'}$

$$f_a(x) = G_a(x) + F_a(x)$$

where

$$\begin{aligned} G_a(x) &:= \sum_{\lambda \in \Lambda^d \times 0^d} d_\lambda(a)\Psi_\lambda(x) \\ F_a(x) &:= \sum_{\lambda \in \Lambda^d \times L^d} d_\lambda(a)\Psi_\lambda(x) \end{aligned}$$

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The trace f_a can be written

$$f_a = \sum_{\lambda\in\Lambda^d\times\{0,1\}^d} d_\lambda(a)\Psi_\lambda(x)$$

for $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \Lambda^d \times \{0,1\}^d$

Theorem (S.Jaffard)

Let $0 < p, s < \infty$. *If* $f \in B^s_{p,\infty}(\mathbb{R}^D)$, *then for Lebesgue-almost all* $a \in \mathbb{R}^{d'}$
$f_a \in \bigcap_{s' < s} B^{s'}_{p,\infty}(\mathbb{R}^d)$.

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Theorem (J-M Aubry, D. Maman, S. Seuret)

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < s - d/p < +\infty$. For almost all f in $B^s_{p,q}(\mathbb{R}^D)$, for Lebesgue-almost all $a \in \mathbb{R}^{d'}$, the following holds:

1. *the spectrum of singularities of f_a is*

$$d_{f_a}(h) = \begin{cases} d + (h-s)p & \text{if } h \in [s - d/p, s] \\ -\infty & \text{else.} \end{cases}$$

2. *for every open set $\Omega \subset \mathbb{R}^d$, the level set $E_{f_a}(s) \cap \Omega$ has full Lebesgue measure in Ω.*

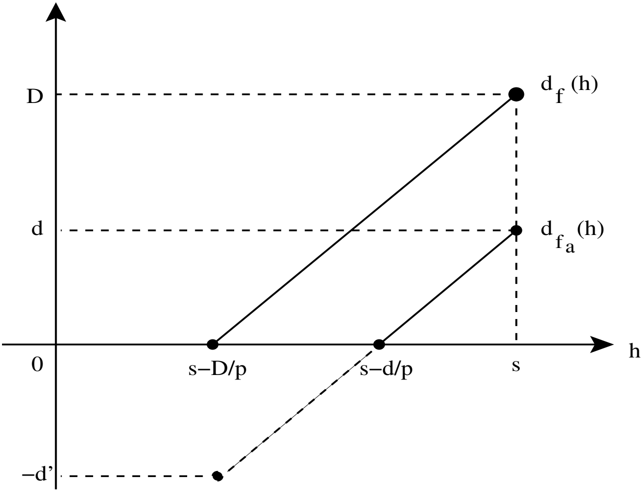

Figure: Singularity spectrum of almost all $f \in B^s_{p,q}(\mathbb{R}^D)$ and its trace f_a for Lebesgue almost every $a \in \mathbb{R}^{d'}$.

Definition

Let $B(x,r)$ denote the closed l^∞ ball of radius r around x in $[0,1]^d$. For $\alpha \geq 1$ and $j \in \mathbb{N}$, let

$$\mathcal{X}_j^\alpha := \bigcup_{k \in \mathbb{Z}_j^d} B(k2^{-j}, 2^{-j\alpha})$$

$$\text{and} \quad \mathcal{X}^\alpha := \limsup_{j \to \infty} \mathcal{X}_j^\alpha$$

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Theorem

There exists a positive σ-finite measure m_α carried by $\mathcal{X}^\alpha$ and such that any set E having Hausdorff dimension $\dim_{\mathcal{H}}(E) < \frac{d}{\alpha}$ has measure $m_\alpha(E) = 0$.
In particular, $m_\alpha(\mathcal{X}^\alpha) > 0$ and $\dim_{\mathcal{H}} \mathcal{X}^\alpha = d/\alpha$.

Definition

Suppose that $0 < s - D/p < \infty$ and $0 < q \leq \infty$. Let $\alpha \geq 1$ and let us define the exponent

$$H(\alpha) := s - \frac{d}{p} + \frac{d}{\alpha p}.$$

and the set

$$\mathcal{F}_\alpha := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \text{ of full Lebesgue measure such that } a \in \mathcal{A}(f) \Longrightarrow \forall x \in \mathcal{X}^\alpha,\ h_{f_a}(x) \leq H(\alpha) \Big\}$$

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Theorem

Suppose that $0 < s - D/p < \infty$ *and* $0 < q \leq \infty$.
The set $\mathcal{F}_\alpha$ *is prevalent in* $B^s_{p,q}([0,1]^D)$.

universally measurable: very delicate
prevalent: technique of the probe space

From now on, let $(\alpha_n)_{n\in\mathbb{N}}$ be a dense sequence in $[1,\infty)$ such that $\alpha_0 = 1$.

Corollary

The set

$$\mathcal{F} := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \text{ of full Lebesgue measure such that } a \in \mathcal{A}(f) \Rightarrow \forall n \in \mathbb{N}, \forall x \in \mathcal{X}^{\alpha_n}, h_{f_a}(x) \leq H(\alpha_n) \Big\}$$

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is prevalent in $B^s_{p,q}([0,1]^D)$.

We apply this corollary with $\alpha_n = \alpha_0 = 1$: if f belongs to the prevalent set $\mathcal{F}$, then for any $a \in \mathcal{A}(f)$, for any $x \in \mathcal{X}^{\alpha_0} = \mathcal{X}^1 = [0,1]^d$, $h_{f_a}(x) \le H(\alpha_0) = s$. Thus :

Proposition

For almost all $f \in B^s_{p,q}([0,1]^D)$, *for Lebesgue-almost all* $a \in [0,1]^{d'}$, *for all* $x \in [0,1]^d$, $h_{f_a}(x) \le s$.

Recall :

Corollary

$$\mathcal{F} := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \textit{ of full Lebesgue measure such that } a \in \mathcal{A}(f) \Rightarrow \forall n \in \mathbb{N}, \forall x \in \mathcal{X}^{\alpha_n}, h_{f_a}(x) \le H(\alpha_n) \Big\}$$

is prevalent in $B^s_{p,q}([0,1]^D)$.

Consider a function f in the prevalent set $\mathcal{F}$. Let $h \in (s - d/p, s]$. This exponent can be written

$$h = H(\alpha) = s - \frac{d}{p} + \frac{d}{\alpha p}$$

Let us assume that $\alpha > 1$, i.e. $H(\alpha) \in (s - d/p, s)$. Consider a subsequence $(\alpha_{\phi(n)})_{n\in\mathbb{N}}$ of $(\alpha_n)_{n\in\mathbb{N}}$ which is nondecreasing and converges to α
Remark that $\mathcal{X}^\alpha \subset \bigcap_{n\ge1} \mathcal{X}^{\alpha_{\phi(n)}} \subset \{x : h_{f_a}(x) \le H(\alpha)\}$.

Let us introduce the set $\mathcal{Y}^\alpha := \big\{x : h_{f_a}(x) < H(\alpha)\big\}$. Clearly,

$$\mathcal{Y}^\alpha = \bigcup_{n\geq 1} \big\{x : h_{f_a}(x) \leq H(\alpha) - 1/n\big\}.$$

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Thanks to result about the upper bound of the spectrum, each set $\left\{x : h_{f_a}(x) \leq H(\alpha) - 1/n\right\}$ has Hausdorff dimension strictly less than d/α.
For the m_α measure defined before, we have : $m_\alpha(\mathcal{X}^\alpha) > 0$ and $m_\alpha(\mathcal{Y}^\alpha) = 0$.
So we have $m_\alpha(\mathcal{X}^\alpha \backslash \mathcal{Y}^\alpha) > 0$.

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For the m_α measure defined before, we have : $m_\alpha(\mathcal{X}^\alpha) > 0$ and $m_\alpha(\mathcal{Y}^\alpha) = 0$.
So we have $m_\alpha(\mathcal{X}^\alpha \backslash \mathcal{Y}^\alpha) > 0$.
This means equivalently that $m_\alpha(\{x \in \mathcal{X}^\alpha : h_{f_a}(x) = H(\alpha)\}) > 0$. This implies that the set $\{x \in \mathcal{X}^\alpha : h_{f_a}(x) = H(\alpha)\}$ has Hausdorff dimension greater than d/α, and thus

$$d_{f_a}(h) = d_{f_a}(H(\alpha)) = \dim_{\mathcal{H}}\{x : h_{f_a}(x) = H(\alpha)\} \geq d/\alpha = p(h-s) + d,$$

the last equality following from the definition of $H(\alpha)$.

Definition

Let $d > 0$. A function $\eta : \mathbb{R}^+ \to \mathbb{R}$ is said to be d-admissible if

$$s(q) = q\eta(1/q)$$

is concave and satisfy $0 \leq s'(q) \leq d$.
It is strongly d-admissible if furthermore $s(0) > 0$.

The following spaces are associated to η:
Let $d < D$, we consider

$$\begin{aligned} V^D &= \bigcap_{\epsilon>0, 0<p<\infty} B_p^{(\eta(p)-\epsilon)/p,p}([0,1]^D). \\ V &= \bigcap_{\epsilon>0, 0<p<\infty} B_p^{(\eta(p)-\epsilon)/p,p}([0,1]^d) \end{aligned}$$

Theorem

Let $0 < d < D$ two fixed integers, and let η strongly d-admissible
For almost all f in V^D, for Lebesgue-almost all $a \in [0,1]^{d'}$, the following holds:

1. $f_a \in V$
2. *The spectrum of singularities of f_a is:*

$$\begin{array}{ll} \text{for every } H \in \left[s(0), \frac{d}{p_c}\right], & d_{f_a}(h) = \inf_{p \geq p_c} (pH - \eta(p) + d), \\ \text{for every } H \notin \left[s(0), \frac{d}{p_c}\right], & E_{f_a}(h) = \emptyset. \end{array}$$

where p_c is the only critical point such that $\eta(p_c) = d$

3. $\forall p > 0 \ \eta_{f_a}(p) = \eta(p)$

where :

$$V^D = \bigcap_{\epsilon > 0, 0 < p < \infty} B_p^{(\eta(p)-\epsilon)/p,p}([0,1]^D).$$

$$V = \bigcap_{\epsilon > 0, 0 < p < \infty} B_p^{(\eta(p)-\epsilon)/p,p}([0,1]^d)$$

Definition (Baire's genericity)

Given a Baire's space E
It is said that a property is hold generically or quasi-all function of E satisfy this property if the set of functions that satisfy contains a countable intersection of everywhere dense open.