Toward a Multifractal Formalism for oscillating singularities

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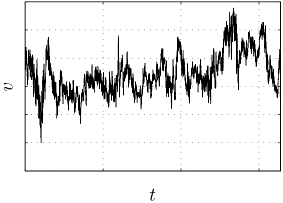
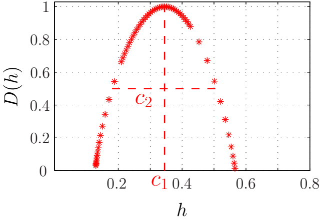
Motivation

Fully developed Turbulence

- Intermittency Phenomenom ⇒ Multifractal Analysis :





Estimation :

$c_1 \simeq 0.345$

$c_2 \simeq 0.0.27$

Intermittency characterized by c_2

- Is there any oscillating singularities?

Outline

Goal : detect oscillating singularities in random field

- Regularity exponent and Multifractal formalism
- Oscillation exponent and fractional integration
- Multifractal formalim for oscillating singularities
- Application to synthetic and experimental data
- Conclusions

Regularity exponent and Multifractal formalism

Regularity exponent

$f(x),\ x \in [0, n)$ *Signal*

- local singularity exponent : the Hölder exponent

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)} \quad h(x_0) \in \mathbb{R}^+$$

$P_N(x - x_0)$ polynomial of order N, $N < h(x_0) < N + 1$.

Example : Cusp singularities, $f(x) = |x - x_0|^h$, $h \in \mathbb{R}$

- $T_f(a, x) \sim a^{h(x)}$, *Multiresolution Coefficients* of f depending on a space parameter x and a scale parameter a.

Multifractal Formalism

- Scale Invariance :

$$M_q(a) = \langle |T_f(a,x)|^q \rangle = F_q a^{\zeta(q)}, \;\; a \in [\eta, L], \; L/\eta \gg 1$$

- Singularity spectrum

$$D(h) = d_H\{x \,|\, h(x) = h\}$$

d_H Haussdorf (or fractal) dimension.

(Parisi and Frisch 1985)

$\Rightarrow$ probability to find h at scale a is $P_a(h) = a^{1-D(h)}$

- Multifractal Formalism

$\zeta(q)$ and $D(h)$ are linked by a Legendre Transform

$$\zeta(q) = \min_h(qh - D(h)) \text{ and } D(h) = \min_q(qh - \zeta(q))$$

Multifractal Formalism

- $M_q(a) = \langle |T(a,x)|^q \rangle \sim a^{\zeta(q)}$
  q linear regressions for $\zeta(q)$
  $+$ Legendre transform for $D(h)$

- $h_q(a) = \langle \hat{T}(a,x) \log |T(a,x)| \rangle \sim a^{h(q)}$
  $D_q(a) = \langle \hat{T}(a,x) \log \hat{T}(a,x) \rangle \sim a^{D(q)}$
  with $\hat{T}(a,x) = |T(a,x)|^q / \sum_x |T(a,x)|^q$
  $2q$ linear regressions

(Muzy *et al* 1994)

- $C_1(a) = \langle \log |T(a,x)| \rangle \sim c_1 \log(a)$
  $C_2(a) = \langle (\log |T(a,x)|)^2 \rangle - \langle \log |T(a,x)| \rangle^2 \sim -c_2 \log(a)$

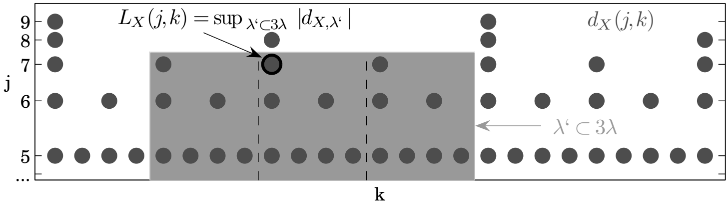
$$\zeta(q) = c_1 q - \frac{c_2}{2} q^2 + \frac{c_3}{6} q^3 + \dots$$

(Castaing *et al* 1993, Delour *et al* 2001)

Only two linear regressions : $c_2 = 0 \Leftrightarrow$ monofractal

Multiresolution coefficients used

- Wavelet coefficients of a Dyadic ($c_f(j,k)$)
  or a continuous Wavelet Transform
- WTMM defined from continuous WT (Arneodo *et al* 1995)
- Wavelet Leaders defined from Dyadic WT : $L_f(j,k)$



(Jaffard *et al*, 2006)

Oscillation exponents and fractional integration

Oscillation exponents

- Oscillation exponent β to describe local oscillations :

  Example : chirp singularities

$$f(x) = |x - x_0|^h \sin(\frac{1}{|x-x_0|^\beta}),\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^*.$$

  h Regularity exponent

- only Wavelet Leaders have the correct behavior :

$$L_f(a,x) \sim a^{h(x)}$$

Fractional Integration

Definition : in Fourier space or in orthogonal wavelet bases

$$I^s[f] = \mathcal{F}^{-1}\left[\int \frac{\mathcal{F}[f](\xi)}{(1+\xi^2)^{s/2}} d\xi\right] \Leftrightarrow \text{replace } c_f(j,k) \text{ by } c_f^s(j,k) = c_f(j,k)/2^{sj}.$$

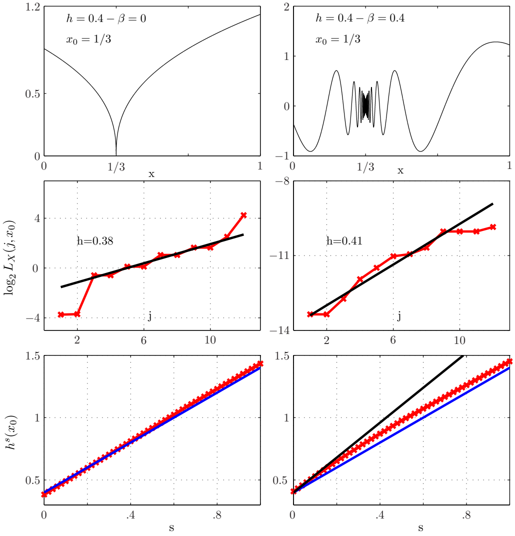
Typical behavior for cusp $h^s(x) = h(x) + s$

for chirps $h^s(x) = h(x) + (1+\beta(x))s$

$$\beta(x) = \lim_{s\to 0} \frac{d h^s(x)}{ds} - 1$$

$\beta(x) = 0$ for cusp

Application to cusp and Chirp



$L(j,k)$ leaders of the signal

$L^s(j,k)$ leaders of the fractionally integrated signal by a factor s

- - - cusp behavior

- - - chirp behavior

—— estimation

Multiresolution coefficients

For all singularities (chirps or cusp)

$$L(j,x) \sim 2^{-h(x)j}$$

$$L^s(j,x) \sim 2^{-h^s(x)j} = 2^{-(h(x)+(1+\beta(x))s)j}.$$

We define a new Multiresolution coefficient, the β-leaders :

$$B^s(j,x) = \frac{1}{2^j}\left(L^s(j,x)/L(j,x)\right)^{1/s} \quad \text{for } s \longrightarrow 0$$

$$\Rightarrow B^s(j,x) \sim 2^{-\beta(x)j}$$

Multifractal Formalism for Oscillating Singularities

- $M_q(j) = \langle |B(j,k)|^q \rangle \sim a^{j\zeta^\beta(q)}$
- $h_q(j) = \langle \hat{B}(j,k) \log |B(j,k)| \rangle \sim 2^{j\beta(q)}$
  $D_q(j) = \langle \hat{B}(j,k) \log \hat{B}(j,k) \rangle \sim a^{jD^\beta(q)}$
  with $\hat{B}(j,k) = |B(j,k)|^q / \sum_k |B(j,k)|^q$
- $C_1(j) = \langle \log |B(j,k)| \rangle \sim < c_1^\beta > \log(2^j)$
  $C_2(j) = \langle (\log |B(j,k)|)^2 \rangle - \langle \log |B(j,k)| \rangle^2 \sim -c_2^\beta \log(2^j)$

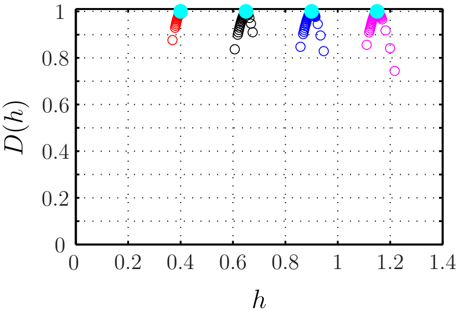
Application to synthetic processes

Fractional Brownian Motion

All estimations are done with 500 realisations of 2^{17} points

Theoretical (•) and estimated $D^S(h^s)$ spectrum
(with Wavelet Leaders)

$c_1 = 0.04$
$c_2 = 10^{-3}$

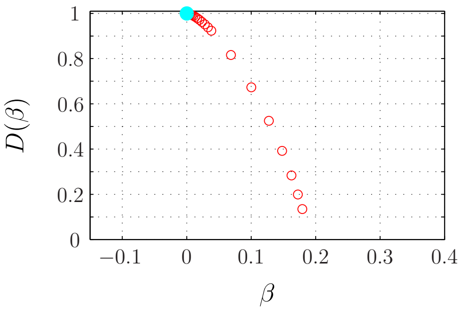


$s = 0$
$s = 0.25$
$s = 0.5$
$s = 0.75$

Translation of speed s

Fractional Brownian Motion

Theoretical (•) and estimated $D(\beta)$ spectrum of oscillation
(with Wavelet β-Leaders; $s = 0.2$)

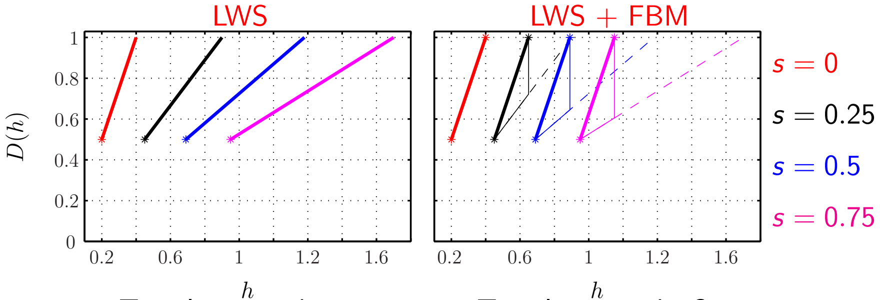


$D(\beta)$ maximum for $\beta = 0 \Rightarrow$ no oscillating singularity.

Lacunary Wavelet Series

With an orthonormal wavelet basis (in d dimension)
On the 2^{dj} wavelet coefficients $c(j, x)$
we choose at random $2^{\gamma j}$ coefficients with value $2^{\alpha j}$
The other coefficients have a null value.

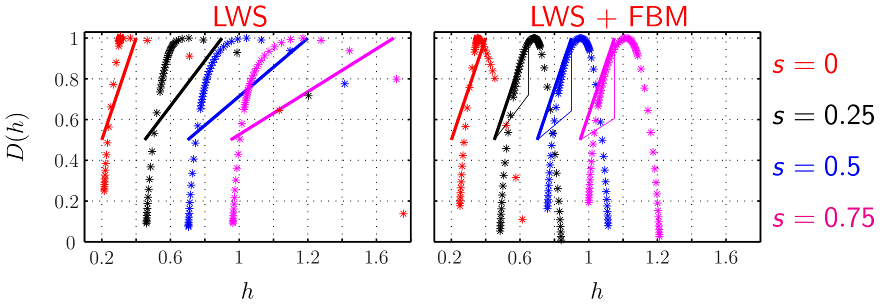
Theoretical $D^S(h^s)$ spectrum ($\alpha = 0.2$, $\gamma = 0.5$)



Travel at speed $> s$ Travel at speed s ?

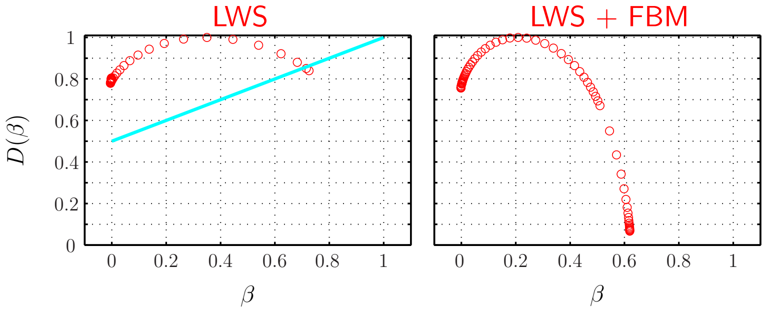
Lacunary Wavelet Series

Theoretical (—) and estimated ($\star$) $D^S(h^s)$ singularity spectrum
(with Wavelet Leaders)



Lacunary Wavelet Series

Theoretical (—) and estimated $D(\beta)$ spectrum of oscillation (with Wavelet β-Leaders; $s = 0.2$)



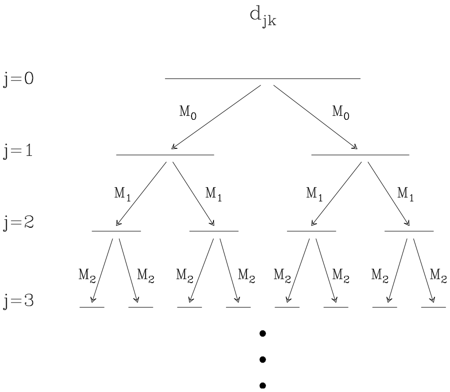
$D(\beta)$ maximum for $\beta > 0$

$\Rightarrow$ detection of oscillating singularities

unsatisfactory estimation

Random Wavelet Cascade & Series

$c(j,k)$ orthonormal wavelet bases

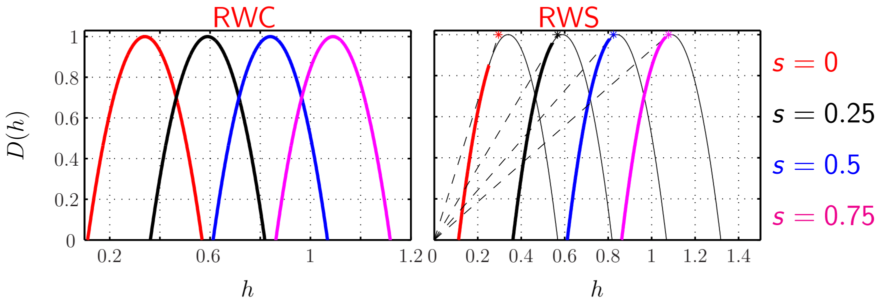


RWC :
$c(j=0,.)$ Gaussian
Multiplicative weight
M_j log-normal for all j.
$(m = c_1, \sigma^2 = c_2)$
Then reconstruction

RWS:
same as RWC
but random shuffling of
$c(j,.)$ for all j
before reconstruction.

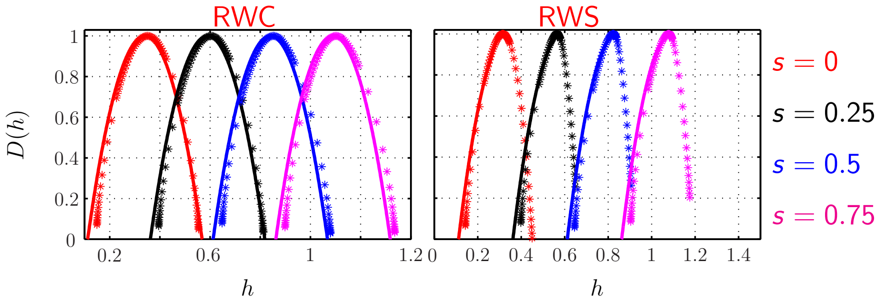
Random Wavelet Cascade & Series

Theoretical $D^S(h^s)$ singularity spectrum
$(c_1 = 0.34,\ c_2 = 0.026)$



Random Wavelet Cascade & Series

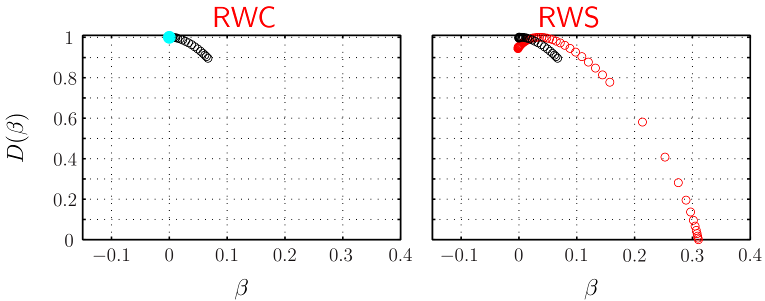
Theoretical and estimated $D^S(h^s)$ singularity spectrum
$(c_1 = 0.34,\ c_2 = 0.026)$
(with Wavelet Leaders)



(estimation : $c_1 = 0.35,\ c_2 = 0.025$)

Random Wavelet Cascade & Series

Theoretical (•) and estimated $D(\beta)$ spectrum of oscillation (with Wavelet β-Leaders; $s = 0.2$)



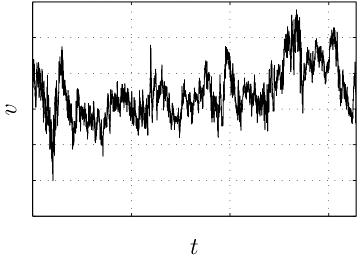
$D(\beta)$ maximum for $\beta = 0$ $\Rightarrow$ no oscillating singularity.

$D(\beta)$ maximum for $\beta > 0$ $\Rightarrow$ oscillating singularities

Fully developed Turbulence

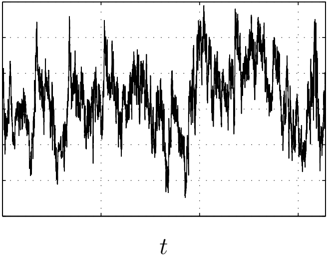
Longitudinal velocity at one location

Wind Tunnel - $R_\lambda \sim 2000$
(Castaing *et al* 1993)



300 realisations of 2^{17} points

Helium Jet - $R_\lambda = 929$
(Chanal *et al* 2000)

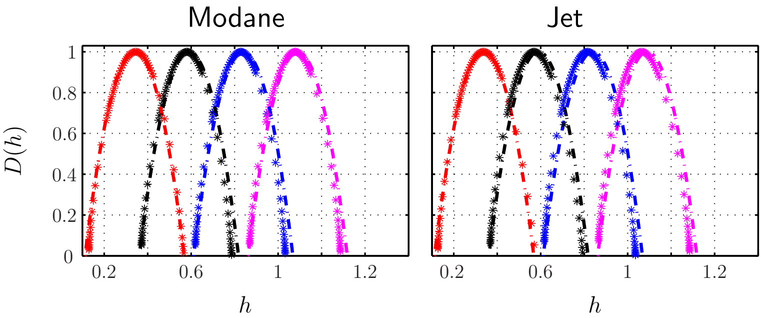


500 realisations of 2^{17} points

Fully developed Turbulence

Estimated ($\star$) $D^S(h^s)$ singularity spectrum
(with Wavelet Leaders)

- - - Theoretical RWC ($c_1 = 0.34$, $c_2 = 0.026$)

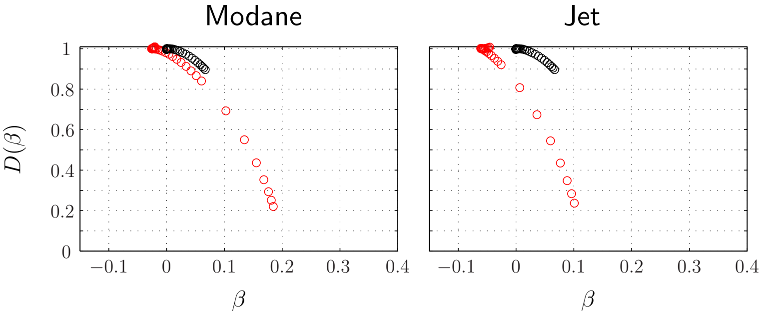


No deformation - translation of speed $\sim s$

Fully developed Turbulence

Estimated (∘) $D(\beta)$ spectrum of oscillation
(with Wavelet β-Leaders; $s = 0.2$)

∘ estimated RWC ($c_1 = 0.34$, $c_2 = 0.026$)



$D(\beta)$ maximum for β slighly negative ⇒ no oscillating singularity.

Conclusions

- allow us to detect the presence of oscillating singularity in multifractal fields
- unsatisfactory estimates of $D(\beta)$
- implemented for 1D or 2D data set
- almost no extra computational cost compare to regular MF
- no oscillating singularity found in turbulence data