Toward a Multifractal Formalism for oscillating singularities

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Motivation

Fully developed Turbulence

- Intermittency Phenomenom $\Rightarrow$ Multifractal Analysis :

![Graph of Intermittency Phenomenon]

- Estimation :
  \[ c_1 \approx 0.345 \]
  \[ c_2 \approx 0.027 \]

Intermittency characterized by \( c_2 \)

- Is there any oscillating singularities?
Goal: detect oscillating singularities in random field

- Regularity exponent and Multifractal formalism
- Oscillation exponent and fractional integration
- Multifractal formalism for oscillating singularities
- Application to synthetic and experimental data
- Conclusions
Regularity exponent and Multifractal formalism

Oscillation exponent and fractional integration

Application to synthetic and experimental data

Conclusions

Regularity exponent
Multifractal formalism

Roux Stéphane

Toward a Multifractal Formalism for oscillating singularities
Regularity exponent

\[ f(x), \ x \in [0, n) \ \text{Signal} \]

- local singularity exponent: the Hölder exponent
  \[ |f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)} \quad h(x_0) \in \mathbb{R}^+ \]
  \[ P_N(x - x_0) \) polynomial of order \( N, N < h(x_0) < N + 1. \]

  Example: Cusp singularities, \( f(x) = |x - x_0|^h, h \in \mathbb{R} \)

- \( T_f(a, x) \sim a^{h(x)}, \) \textit{Multiresolution Coefficients of } \( f \)
  depending on a space parameter \( x \)
  and a scale parameter \( a. \)
Multifractal Formalism

- **Scale Invariance:**
  \[ M_q(a) = \langle |T_f(a,x)|^q \rangle = F_q a^{\zeta(q)}, \ a \in [\eta, L], \ L/\eta \gg 1 \]

- **Singularity spectrum**
  \[ D(h) = d_H \{ x \mid h(x) = h \} \]
  \(d_H\) Haussdorf (or fractal) dimension.
  (Parisi and Frisch 1985)
  \( \Rightarrow \) probability to find \( h \) at scale \( a \) is \( P_a(h) = a^{1-D(h)} \)

- **Multifractal Formalism**
  \( \zeta(q) \) and \( D(h) \) are linked by a Legendre Transform
  \[ \zeta(q) = \min_h (qh - D(h)) \text{ and } D(h) = \min_q (qh - \zeta(q)) \]
Multifractal Formalism

- \( M_q(a) = \langle |T(a, x)|^q \rangle \sim a^{\zeta(q)} \)
  - \( q \) linear regressions for \( \zeta(q) \)
  - + Legendre transform for \( D(h) \)

- \( h_q(a) = \langle \hat{T}(a, x) \log |T(a, x)| \rangle \sim a^{h(q)} \)
  - \( D_q(a) = \langle \hat{T}(a, x) \log \hat{T}(a, x) \rangle \sim a^{D(q)} \)
  - with \( \hat{T}(a, x) = |T(a, x)|^q / \sum_x |T(a, x)|^q \)
  - 2\( q \) linear regressions

(Muzy et al 1994)

- \( C_1(a) = \langle \log |T(a, x)| \rangle \sim c_1 \log(a) \)
  - \( C_2(a) = \langle (\log |T(a, x)|)^2 \rangle - \langle \log |T(a, x)| \rangle^2 \sim -c_2 \log(a) \)

\[ \zeta(q) = c_1 q - \frac{c_2}{2} q^2 + \frac{c_3}{6} q^3 + ... \]

(Castaing et al 1993, Delour et al 2001)

Only two linear regressions : \( c_2 = 0 \Leftrightarrow \) monofractal
Multiresolution coefficients used

- Wavelet coefficients of a Dyadic \( c_f(j, k) \)
  or a continuous Wavelet Transform
- WTMM defined from continuous WT \( (\text{Arneodo et al 1995}) \)
- Wavelet Leaders defined from Dyadic WT: \( L_f(j, k) \)

\[
L_X(j, k) = \sup_{\lambda \subseteq 3\lambda} |d_{X,\lambda}|
\]

\( \lambda^i \subset 3\lambda \) 

\( (\text{Jaffard et al, 2006}) \)
Oscillation exponents and fractional integration
Oscillation exponent $\beta$ to describe local oscillations:

Example: chirp singularities

$$f(x) = |x - x_0|^h \sin \left( \frac{1}{|x - x_0|^{\beta}} \right), \alpha \in \mathbb{R}, \beta \in \mathbb{R}^*.$$  

$h$ Regularity exponent

only Wavelet Leaders have the correct behavior:

$$L_f(a, x) \sim a^{h(x)}$$
Fractional Integration

Definition: in Fourier space or in orthogonal wavelet bases

\[ I^s[f] = \mathcal{F}^{-1} \left[ \int \frac{\mathcal{F}[f](\xi)}{(1+\xi^2)^{s/2}} d\xi \right] \iff \text{replace } c_f(j, k) \text{ by } c_f^s(j, k) = c_f(j, k) / 2^{sj}. \]

Typical behavior for cusp

\[ h^s(x) = h(x) + s \]

for chirps

\[ h^s(x) = h(x) + (1 + \beta(x))s \]

\[ \beta(x) = \lim_{s \to 0} \frac{dh^s(x)}{ds} - 1 \]

\[ \beta(x) = 0 \text{ for cusp} \]
Application to cusp and Chirp

$L(j, k)$ leaders of the signal

$L^s(j, k)$ leaders of the fractionally integrated signal by a factor $s$

- - - cusp behavior

- - - chirp behavior

--- estimation
Multiresolution coefficients

For all singularities (chirps or cusp)

\[ L(j, x) \sim 2^{-h(x)j} \]

\[ L^s(j, x) \sim 2^{-h^s(x)j} = 2^{-(h(x)+(1+\beta(x))s)j} \]

We define a new Multiresolution coefficient, the \( \beta \)-leaders:

\[ B^s(j, x) = \frac{1}{2^j} \left( \frac{L^s(j, x)}{L(j, x)} \right)^{1/s} \text{ for } s \to 0 \]

\[ \Rightarrow B^s(j, x) \sim 2^{-\beta(x)j} \]
Multifractal Formalism for Oscillating Singularities

- $M_q(j) = \langle |B(j, k)|^q \rangle \sim a^j \xi(q)$

- $h_q(j) = \langle \hat{B}(j, k) \log |B(j, k)| \rangle \sim 2^j \beta(q)$
- $D_q(j) = \langle \hat{B}(j, k) \log \hat{B}(j, k) \rangle \sim a^j D \beta(q)$

with $\hat{B}(j, k) = |B(j, k)|^q / \sum_k |B(j, k)|^q$

- $C_1(j) = \langle \log |B(j, k)| \rangle \sim < c_1^\beta > \log(2^j)$
- $C_2(j) = \langle (\log |B(j, k)|)^2 \rangle - \langle \log |B(j, k)| \rangle^2 \sim -c_2^\beta \log(2^j)$
Application to synthetic processes
All estimations are done with 500 realisations of $2^{17}$ points.

Theoretical (●) and estimated $D^S(h^s)$ spectrum (with Wavelet Leaders)

$c_1 = 0.04$
$c_2 = 10^{-3}$

$s = 0$
$s = 0.25$
$s = 0.5$
$s = 0.75$

Translation of speed $s$
The theoretical (●) and estimated $D(\beta)$ spectrum of oscillation (with Wavelet $\beta$-Leaders; $s = 0.2$)

$D(\beta)$ maximum for $\beta = 0 \Rightarrow$ no oscillating singularity.
Lacunary Wavelet Series

With an orthonormal wavelet basis (in $d$ dimension)
On the $2^{d^j}$ wavelet coefficients $c(j, x)$
we choose at random $2^{\gamma j}$ coefficients with value $2^{\alpha j}$
The other coefficients have a null value.

Theoretical $D^S(h^s)$ spectrum ($\alpha = 0.2, \gamma = 0.5$)

Travel at speed $h > s$  
Travel at speed $s$?
Lacunary Wavelet Series

Theoretical (—) and estimated (⋆) $D^S(h^s)$ singularity spectrum (with Wavelet Leaders)

\[ D(h) \]

LWS

LWS + FBM

$s = 0$

$s = 0.25$

$s = 0.5$

$s = 0.75$
Lacunary Wavelet Series

Theoretical (—) and estimated $D(\beta)$ spectrum of oscillation (with Wavelet $\beta$-Leaders; $s = 0.2$)

$LWS$

$LWS + FBM$

$D(\beta)$ maximum for $\beta > 0$

$\Rightarrow$ detection of oscillating singularities

unsatisfactory estimation
Random Wavelet Cascade & Series

c\(j, k\) orthonormal wavelet bases

d_{jk}

\[\begin{align*}
  j=0 & \quad \text{M} & \quad \text{M} \\
  j=1 & \quad \text{M} & \quad \text{M} \\
  j=2 & \quad \text{M} & \quad \text{M} & \quad \text{M} \\
  j=3 & \quad \text{M} & \quad \text{M} & \quad \text{M} & \quad \text{M} \\
  & \ldots & \ldots & \ldots & \ldots & \ldots
\end{align*}\]

RWC:
c(0, .) Gaussian
Multiplicative weight
\(M_j\) log-normal for all \(j\).
\((m = c_1, \sigma^2 = c_2)\)
Then reconstruction

RWS:
same as RWC
but random shuffling of
c(0, .) for all \(j\)
before reconstruction.
Random Wavelet Cascade & Series

Theoretical $D^S(h^s)$ singularity spectrum
$(c_1 = 0.34, c_2 = 0.026)$
Theoretical and estimated $D^S(h^s)$ singularity spectrum

$(c_1 = 0.34, c_2 = 0.026)$

(with Wavelet Leaders)

( estimation : $c_1 = 0.35, c_2 = 0.025$)
Theoretical (○) and estimated $D(\beta)$ spectrum of oscillation (with Wavelet $\beta$-Leaders; $s = 0.2$)

- $D(\beta)$ maximum for $\beta = 0$ ⇒ no oscillating singularity.
- $D(\beta)$ maximum for $\beta > 0$ ⇒ oscillating singularities
Longitudinal velocity at one location

Wind Tunnel - $R_\lambda \sim 2000$
(Castaing et al. 1993)

Helium Jet - $R_\lambda = 929$
(Chanal et al. 2000)

300 realisations of $2^{17}$ points

500 realisations of $2^{17}$ points
Fully developed Turbulence

Estimated (⋆) $D^S(h^s)$ singularity spectrum
(with Wavelet Leaders)

- - - Theoretical RWC ($c_1 = 0.34$, $c_2 = 0.026$)

No deformation - translation of speed $\sim s$
Estimated (○) $D(\beta)$ spectrum of oscillation
(with Wavelet $\beta$-Leaders; $s = 0.2$)

○ estimated RWC ($c_1 = 0.34$, $c_2 = 0.026$)

$D(\beta)$ maximum for $\beta$ slightly negative $\Rightarrow$ no oscillating singularity.
Conclusions

- allow us to detect the presence of oscillating singularity in multifractal fields
- unsatisfactory estimates of $D(\beta)$
- implemented for 1D or 2D data set
- almost no extra computational cost compare to regular MF
- no oscillating singularity found in turbulence data